Second Year Report

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1
Introduction

§ 1. Introduction

§ 1.1 Physics

Percolation has been introduced and developed in the 1950s. (Hammersley et. al., 1954; Broadbent et. al., 1957) A typical physical problem which gives rise to problems in percolation is that of finding the probability that the centre of a sufficiently large porous stone gets wet when immersed in water. The internal structure of such a stone can be viewed as comprised of pores and solids. Water can seep through clusters of pores but is blocked when solids are encountered. Therefore the water can only reach the centre of the stone if there exists an open path, in other words a single cluster of pores, which leads it there. When the size of the rock is sufficiently large, this cluster is called an infinite cluster.

If on the other hand we consider the solids instead of the voids, we can reason that since the stone is rigid there have to exist at least one infinite cluster of solids. Then, because there must be rocks of various degrees of porosity which allow water to pass through, there must be a range of porosity which at one end is a nonpermeable stone where there exists an infinite cluster of solids but not an infinite cluster of pores, while at the other end it is the other way round, that is there exists an infinite cluster of pores but not one of solids. In other words, the latter case is the case where the rock has disintegrated into pieces. If one considers the pores and the solids as being two phases opposite to each other, then one can reason that the total volume of the two stays the same, whereas the ratio between them could vary.

The study of blockages in porous media can be translated into the study of percolation on the media when pores randomly turn into solids. In other words, it is the study of the formation of an infinite cluster of one phase within the infinite mass of the other. More often the latter infinite mass is taken for granted as always exists and remains the same. The domain of consideration is thus reduced to only the original pore spaces. Then, the study becomes that of inversion between two phases, namely the infinite cluster of free pores and the finite clusters of free pores. While the existence of an infinite cluster or finite clusters of blockages is of no consequences when one is only interested in the percolation point, it is considerably important when one wants to contemplate on the behaviours on either side of that critical point. It is important also when one studies changes in the rate of flow through the media.

One example is the study of traffic networks. Traffic congestion can be described as three degrees of flowability: free-flowing, congested, and stand-still. In a free-flowing traffic there are infinite clusters of roads and finite clusters of blocked roads. In a congested traffic there are both infinite clusters of roads and infinite clusters of blocked roads. In a stand-still traffic there are no infinite clusters of roads, while there are infinite clusters of blocked roads.

Applications of the percolation theory includes the study of forest fires and epidemics, the study of defects in semiconductors, the study of the effective resistance of a disordered mixture of two materials, the study of the Curie point of spontaneous magnetisation in ferromagnets and the Ising model, and the study of the rigidity of networks.

Phenomena which contain phase transitions are normally related to percolation theory. Examples of physical phase transition includes the boiling point and the point of evaporation, the transition of materials into superconductivity state, the triggering of a chain reaction in nuclear fission, and the triggering of nuclear fusion of hydrogen burning in a new-born star. Apart from these, there are philosophical phase transitions which include the nature of understanding, the nature of scientific discoveries and technological progress, the making of mobs, revolutions and wars, enlightenment, and, of course, love. Nature is essentially made up of numerous local linear or gradual relationships bound together in a larger scale by abrupt transitions.

The word *percolation* is derived from the word *percolator*, a coffee making machine. The connotation is therefore that of fluid flowing through a porous medium. In dynamic and everyday applications the emphasis is usually on the fluid part, and one studies the point of transition when the flow becomes blocked. There is another side of the same system where one studies the structure of underlying network. In that case the emphasis is on the solid part, and one is interested to find the point where particles percolate to form a solid structure.

Thus the study of percolation falls broadly into two branches, the continuum percolation and the network percolation. The former finds applications in *crystallisation* where one would like to know the number density of crystals that will make, for example, a synthetic zeolite membrane; in *etching of metallic surface* where one would like to find the estimates of the shapes and sizes of the etch pits which overlap and form clusters according to etching time and concentration of the acid; or

in *telecommunication networks* where one would like to determine the number of mobile phone cells which percolates a city. The latter finds applications in *filtration* where one would like to discard filters just before they become blocked or in *traffic networks* where one could compare the robustness to standstill between two traffic network configurations.

Phase transition occurs in various areas. Wherever it is seen, it has the same characteristic of an intensely exponential change in a short period within a long stretch of a much more gradual one. The rate of change at the point of transition is generally astronomical. The sudden switch from one phase to another found in the study of percolation is a phase transition. The assumption of the existence and occurence of infinite clusters as an explanation for these transitions can be justified by considering the promptness and the scale of theoretical transitions in a cosmological setting, namely the turning on of the general relativity at Planck time 10^{-43} seconds after the bigbang, the Grand Unified phase transition at 10^{-35} seconds and the start of nucleosynthesis at 1 second. (cf. Croswell, 1995) The temperature of the universe decreases exponentially and its trend against time has to be plotted on a log-log scale to be comprehensible for otherwise it would lie almost on top of the coordinate axes. Such a plot as shown in Figure ℵ, produced from the data from Vaas (2002), approximates a straight line with negative slope. The percolation at the big bang could possibly be a phase change between matter and antimatter.

There is an idea in cosmology which could easily be as old as the oldest known religion, that describes the universe as alternating in cycles of destruction and creation. One current theory holding that idea tries to explain the big bang as the collision between two universes in five dimensional space-time. My idea is that the universe undergoes phase changes periodically and percolates at the point the same one of which is differently called the big bang, the big crunch, or the big bounce depending how one looks at it. What we see now is only one side of the coin. We can not calculate pass the singularity at the big bang using our present theories because it is another phase complementary to us. That is why that creative singularity baffles us. But I have no doubt that a theory exists which can explain both or, if there are more than two, all phases as well as the transition point. And when we finally discover it we will understand the cosmic history and, hopefully, percolation.

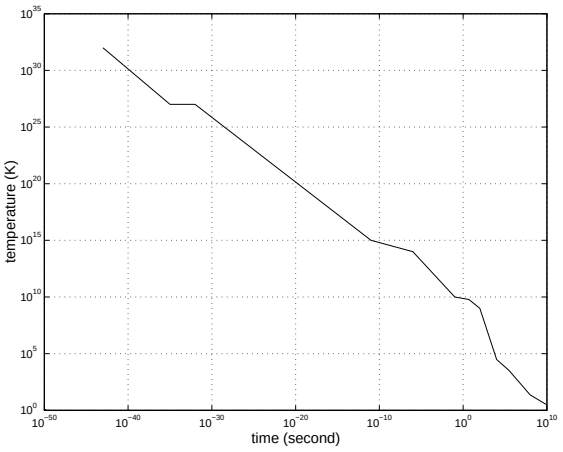


Figure 1.1 Temperature of the universe according to the Big Bang theory.

In an asymmetric membrane the top dense layer determines the transport rate while the porous sublayer acts as a support. Because the permeation rate is inversely proportional to the thickness of the transport barrier, asymmetric membranes show a much higher permeation rate than homogeneous symmetric membranes of the same thickness.

Soap is solution of a sodium salt of a fatty acid, for example sodium stearate ($C_{17}H_{35}COO^-Na^+$.) The films of soap bubbles are made up of a monomolecular layer of amphipathic ions, which have a hydrophilic and a hydrophobic parts. The former forms a hydrophilic polar carboxyl head, which in the case of sodium stearate is the COO^-. The latter forms a tail of hydrocarbon chain, in this case $C_{17}H_{35}$.

	Laboratorical scale		Industrial scale	
Process	origin	application	origin	application
microfiltration	Germany, 1920	bacteria filter		
ultrafiltration	Germany, 1930	laboratory use	America, 1960	concentration of macromolecules
hemodialysis	Netherland, 1950	artificial kidney		
electrodialysis			America, 1955	desalination
hyperfiltration			America, 1960	sea water desalination
gas separation			America, 1979	hydrogen recovery
membrane distil -lation			Germany, 1981	concentration of aqueous solutions
pervaporation			Germany and Netherlands, 1982	dehydration of organic solvents

Table 1.1 *Origin of membrane processes.*

There are three categories of synthetic membranes: porous, nonporous, and carrier. Micro- and ultrafiltration use porous membranes while nonporous membranes are used for dialysis, vapour permeation, gas separation and pervaporation. vAll membranes are required to be mechanically, thermally, and chemically stable. The means of separation, however, varies. For porous membranes it is the pore dimension, for nonporous membranes the difference in diffusivity or solubility, and for carrier membranes the properties of carrier molecules.

Methods	**pore size (μm)**	**porosity**	**membrane types**	**examples**
coating				
phase inversion				
sintering	0.1–10	10–20%	porous	
stretching	0.1–3	90% max	porous	
leaching	0.005 min		porous	
etching	0.02–10	10% max	porous	

Table 1.2 *Methods of preparation of membranes.*

In cosmology, the structure of the Abell Clusters shows a nonrandom distribution which suggests the existence of a second-order grouping, the clusters of clusters of galaxies (Abell, 1958). Apart from the Milky Way and the Andromeda galaxies (M31 or NGC224) there are at least 44 other galaxies in our Local Cluster. The Local Cluster is on the edge of the Local Supercluster which is flattened in shape and the approximate centre of which is at the Virgo Cluster. The Local Supercluster is separated by a nonspherical void of low galaxy density from the Pisces-Perseus supercluster. The latter lies nearly perpendicular to our line of sight, has the shape of a linear filament, and spans over 90° across the sky. The study of superclusters of galaxies often requires the use of superclusters of computers.

Abell (1958) plotted the distribution of the clusters in what he called, the 'Aitoff equal area projection'. By this he probably means the Hammer, aka Hammer-Aitoff, projection because that is what is normally used for astronomical maps in galactic coordinates as well as for maps of the whole world. Inspired by the Aitoff projection, it was created by H. H. Ernst von Hammer as a modification of the Lambert Azimuthal Equal Area projection. The Hammer projection is equal area while the Aitoff projection is not. Both are similar to each other in that both are modified azimuthal projections where the central meridian is a straight line half the length of the equator, the only point free from distortion is the centre point, and there is a moderate distortion throughout. My subsequent investigations lead me to conclude that it is a normal practice in Astronomy and Cosmology to call the Hammer projection as the Aitoff projection.

Powles and Quirke (1984) analyse the numerical trajectory of a molecule in liquid circles by using an empirical fractal parameter called the *Richardson coefficient*, α. In their simulation they used the Lennard-Jones (12,6) intermolecular potential together with a reduced density and pressure. For the length of molecular trajectories in an argonlike liquid they found $\alpha = 0.65$, in comparison with a similar analysis done on a randomised Koch curve of order 6 which gives $\alpha = 0.25$ and the exact $\alpha = 0.2618\ldots$ for K_∞, the Koch curve of an infinite order. The length of a fractal curve is $L(\epsilon) \propto \epsilon^{-\alpha}$, where ϵ is the step distance or scale. The graph between $\log(L(\epsilon)/\sigma)$ and $-\log(\epsilon/2)$ is a straight line with a positive slope represented by the equation $\log(L(\epsilon)/\sigma) = -\alpha \log(\epsilon/\sigma) + K$ where $K = \log k - (1+\alpha)\log 2$, k being a constant. The Leonard-Jones (n,6) potential is the Van der Waals potential between two atoms which is described in the form $V(r) = C_n/r^n - C^6/r^6$ where

C_n and C_6 are constants, and r is the distance between the centres of the two atoms. Similarly, the Leonard-Jones (12,6) potential is $V(r) = 4\epsilon\left((\sigma/r)^{12} - (\sigma/r)^6\right)$. It is sometimes written as $V(r) = \epsilon\left((R/r)^{12} - 2(R/r)^6\right)$, where $R = R_i + R_j$, $\epsilon = \sqrt{\epsilon_i \epsilon_j}$, R_i and ϵ_i being the radius and respactively the interaction energy of each atom. In other words, in the last formula ϵ is the geometrical mean of the interaction energy of each atom while R is the atomic cross-section which is a property of a pair of atoms.

§ 1.2 Cosmological structure

The Lorentz transformation was first written down by Voigt in 1887 as $x' = x - vt$, $y' = \frac{y}{g}$, $z' = \frac{z}{g}$, and $t' = t - \frac{vx}{c^2}$, then by Larmor in 1898, and finally by Lorentz in 1899. Poincaré stated in 1898 on the measure of time that there is no absolute equality of two time intervals. He named the Lorentz transformation after Lorentz and showed that together with the rotations they form a group. Working on the transformations, Einstein announced the special relativity in 1905 as a theory merited by its simplicity and beauty rather than being an explanation of experimental results. For the special relativity and the Lorentz transformations Minkowski founded a four dimensional non-Euclidean space to represent space-time which Einstein adopted used as a basis for his general relativity which appeared in 1915.

2
Division of space

§ 2. Division of space

§ 2.1 Stereographic projection

Stereographic projection is used to represent a three-dimensional figure in two dimensions. It is useful in the study of symmetry of crystals. The stereographic projection of a cube is shown in Figure 2.1.

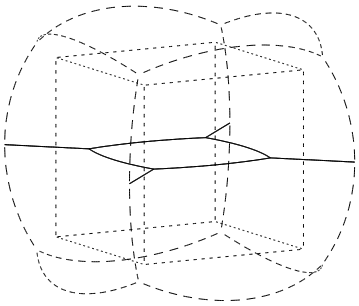


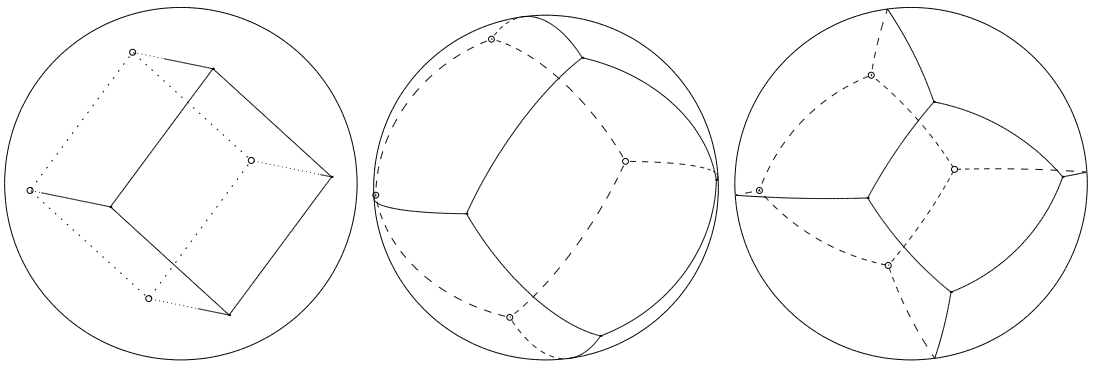
Figure *fste Stereographic projection* of a cube. A stereographic projection is drawn by first projecting each vertex on to the surface of a sphere encircling the polyhedron along the line coming from the origin of the sphere. The projected point on the sphere surface is then projected on to the plane $z = 0$ (ie. (x, y, o)) along the line which originates from it and goes towards the nadir point relative to the half-sphere surrounding it that rests on the $z = 0$ plane.

With this kind of projection, points that are symmetrical to each other with respect to the plane $z = 0$ are projected on to the same point on that plane. Therefore it is a normal practice to distinguish points of the two hemispheres by drawing those in one of them as dots, while drawing the rest as circles. Continuous lines going from one hemisphere to another will become discontinuous on the projected plane. In Figure 2.1, the top and the bottom squares of the cube is projected on top of each other, while the four edges parallel to the z-axis go to the circumference of the great circle of the projected plane first, then retrace their ways back to their vertices.

Figure 2.1 shows that, in effect, what the stereographical projection does is to bloat a polyhedron out into a spherical balloon, and then project the image obtained on the balloon on to the horizontal plane. What the second projection does is to look at the hemisphere above it with a 90° wide-angled lens from the nadir position. In order to compare the shape of the original polyhedron with the top-view of the balloon as well as the stereographical image, Figure 2.2 draws our cube rotated one radian around the vector $(0.3, 0.4, 0.866)$. Such rotation can be done by using the transformation matrix

$$M = \begin{bmatrix} 1-2(y^2+z^2) & 2(xy-wz) & 2(xz+wy) \\ 2(xy+wz) & 1-2(x^2+z^2) & 2(yz-wx) \\ 2(xz-wy) & 2(yz+wx) & 1-2(x^2+y^2) \end{bmatrix},$$

where $[(x, y, z), w] = q$, a quaternion, $(x, y, z) = u \sin\alpha$ and $w = \cos\alpha$. Here $u = (a, b, c)$ is the direction cosine vector of the axis of rotation and $\theta = 2\alpha$ is the angle of rotation.



(a) (b) (c)

Figure 2.2 *Rotated cube, (a) top-view, (b) top-view of its bloated sphere and (c) the stereographic projection. While (c) shows the same amount of symmetry information as the other two, it incorporates more information regarding the z-coordinate than the others, (b) being the least informative in this respect.*

The information regarding symmetries of the polyhedron is preserved by the projection. In practice, when it is applicable and possible, the sphere is positioned such that its centre coincides with the centre of symmetry of the polyhedron, the plane $z = 0$ on one of the symmetry planes, and the planes $y = 0$ and $x = 0$ on two others. Thus Figure the projection of 2.2 clearly shows the

symmetry around the centre as well as the reflection symmetries with respect to the planes $z = 0$, $y = 0$, $x = 0$, $x = y$ and $x = -y$. Figure 2.2, on the other hand, only shows the symmetry around the centre.

§ 2.2 Covering lattices

A covering lattice of any two-dimensional lattice is the lattice obtained by joining midpoints of consecutive edges together. The code in § 7.7 finds covering lattices up to the eighth one and computes the total area of the cells for each case.

The square lattice is the only regular covering lattice, that is it is both the dual and the covering lattices of itself.

§ 2.3 Viscous fingers

Viscous fingering is a happening which occurs when a low viscosity fluid with high pressure penetrates the border of a higher viscosity fluid in a form of thin branching fingers resembling a flash of lightnings in shape. An experimental account as well as descriptive pictures and diagrams are given by Nittmann et al (1985). However, a reference made to one previous work (Hele-Shaw, 1898) was inaccurate in at least three ways, one of which is in the initials of its author, while the rest in the actual contents. The first one can easily be verified with a little research, is that the initials of the author of the paper is H. S., not J. S. S. as written there, the full name being Henry Selby Hele-Shaw (1854–1941). For the remaining two, firstly the paper by Hele-Shaw is about fluid flow pass free boundary of solid, not *viscous fingering instability.* Secondly, Hele-Shaw was an engineer who has been a professor at University college in Liverpool for 17 years, whose interest was in *layer parallel motion* (laminar) and *sinuous* (turbulent) flows pass various bodies of uniform cross section as well as in flows through channels of varying cross section. Born at Billericay, Essex, and a holder of Whitworth scholarships while a student at University college, Bristol, he was eleced to the Royal Society in 1899 because of his experiments done on streamline flow. He has successfully introduced the use of air bubbles in experiments to help portray the stream lines of the flow; his cited work mentions neither the physics of nor the application to petroleum science as the context would have us believe. However, his other and subsequent work (Hele-Shaw, 1899) does mention briefly about the importance of viscosity in nature and the difficulty of modelling the motion of viscous fluids mathematically, but nowhere in either one of these two papers did he consider the interaction between fluids of different viscosity.

§ 2.4 Crystals, quasicrystals, and polycrystals

Crystals, quasicrystals, and polycrystals

The growth process of both crystals and quasicrystals are nonlocal in nature, but that of polycrystals is of a multigrain growth. Crystals with a very large unit cell exists. Quasicrystals cannot grow in the same manner as playing a jig-saw puzzle (Penrose, 1989) because no matter how many steps one looks ahead there will come a point where there is a gap that none of the available basic building blocks can fit in. A *legal* tiling contains no gaps or overlaps; if it can be extended to cover the whole plane it is also a *correct* tiling. A [mistake] occurs when a tile added to a correct tiling renders it a legal but incorrect one. By using the basic units to build larger units having the same structure as these basic units and then recursively repeat the process, one can in the end reach a correct tiling. But this is also a nonlocal process since one still requires the ability to look ahead in order to make sure that the units are always structurally the same.

The fact that zeolites are crystalline rules out the randomness assumption that one may be tempted to use to model their growths as random continuum percolation. One needs to resort to the methods and tools used by geologists if one wants to study zeolites, synthetic as well as natural ones.

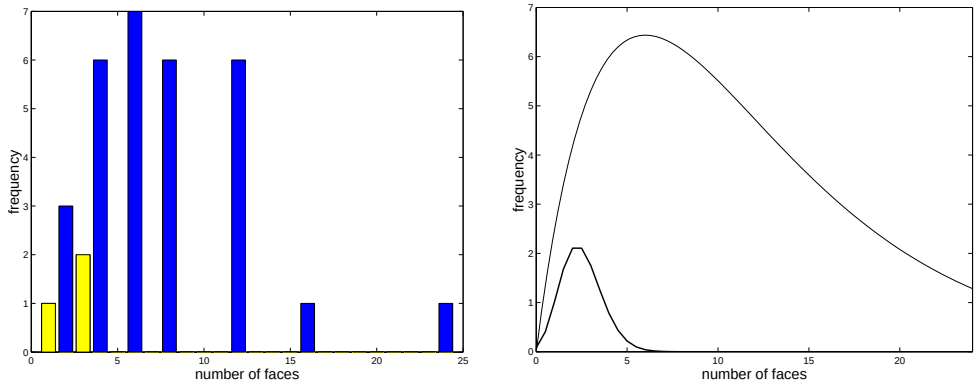
Krýstallos is the Greek word for ice. Crystallography is nowaday the science of the crystalline state which, apart from the crystals themselves, includes such seemingly unlikely materials as plastics, rubber, silk, wool, liquids and gases (*cf* Phillips, 1949). There are four kinds of symmetry axes in crystal models, corresponding to $n = 2$ to 6 in the formula for the angles $\frac{360^\circ}{n}$ by the rotation of which a crystal will repeat itself. These axes are respectively called *diad*, *triad*, *tetrad*, and *hexad*.

Crystals	*centres of symmetry*	*planes of symmetry*	*diad*	*triad*	*tetrad*	*hexad*
Cube	1	9	6	4	3	
Rhombohedron	1	3	3	1		

Table 2.1 *Symmetries of crystals*

The symmetries of the seven crystal systems, the Triclinic contains no axes of symmetry, Monoclinic one diad axis, Orthorhombic three diad axes, Tetragonal one tetrad ais, Cubic four triad axes, Trigonal one triad axis, and Hexagonal one hexad axis. A *monoclinic* crystal has all three axes unequal and one oblique intersection while an *anorthic*, aka *triclinic*, crystal has all three axes unequal and intersecting at oblique angles. The relative development of crystals in different forms give rise to the *habit* of crystal. Thus if one gradually truncates the coigns of a cube one will come to an octahedron, and vice versa. All the continuous transformations between the cube and the octahedron are called *cubo-octahedron*. Crystal models of different habits readily yield on inspection the same symmetry group, but real crystals often look irregular and determination of the symmetry is based on the law of consistancy of angle and uses goniometers.

The number of faces in crystals are usually even numbers. Take for example the 33 non isometric crystal forms (Klein and Hurlbut, 1993). Here only three have their faces in odd number, *viz.* pedion which has only one face, and trigonal prism and trigonal pyramid which have three faces each. For the rest, those which have two faces are pinacoid, dome and sphenoid; four faces rhombic prism, tetragonal prism, rhombic pyramid, tetragonal pyramid, rhombic disphenoid and tetragonal disphenoid; six faces ditrigonal prism, hexagonal prism, ditrigonal pyramid, hexagonal pyramid, trigonal dipyramid, trigonal trapezohedron and rhombohedron; eight faces ditetragonal prism, ditetragonal pyramid, rhombic dipyramid, tetragonal dipyramid, tetragonal trapezohedron and tetragonal scalenohedron; twelve faces dihexagonal prism, dihexagonal pyramid, ditrigonal dipyramid, hexagonal dipyramid, hexagonal trapezohedron and hexagonal scalenohedron; sixteen faces ditetragonal dipyramid; and twenty-four faces dihexagonal dipyramid. Among these, there is none which has its faces in a number of 5, 7, 9–11, 13–15, 23, or 25 and above. In Figure 2.1 the distribution curve for even numbers of faces is a scaled Chi-square distribution $y = K \cdot (x/3)^{(\nu-2)/2} \exp(-(x/3)/2)/(2^{\nu/2}\Gamma(\nu/2))$, where the degree of freedom ν is four and $K = 35$, while the curve for odd numbers of faces is the contour of a scaled Poisson distribution $y = K \cdot \lambda^{2x} \exp(-\lambda)/(2x)!$, where $K = 12$, $\lambda = 5$, and $2x$ is any positive integer or zero.



(a) (b)

Figure 2.1 *(a) number of faces of the thirty-three non isometric crystal forms; (b) approximation curves, the heavy line encompasses* $y = 12\exp(-5) \cdot 5^{2x}/(2x!)$ *while the lighter line* $y = (35/12)x\exp(-x/6)$.

3
Percolation

§ 3. Percolation

The 1970's and the 1980's saw a proliferation of variations on the theme of percolation. Every year there seemed to be a new percolation problem or two. For a *Bethe lattice* Chalupa *et al* (1979) reported a *bootstrap percolation* where those randomly occupied sites with less than m occupied neighbours are recursively emptied one by one until a stable configuration is reached. The problem they are interested in is that where the impurity concentration, dilution and crystal-field interaction compete in magnetic materials compete against the exchange interaction, resulting in the magnetic moments and consequently the magnetic order being destroyed.

Wilkinson and Willemsen (1983) introduced *invasion percolation*. Working with Schlumberger they were interested in the real problem in the oil industry where water displaces oil via capillary action. Their approach was that of a constant flow rate, not one of a constant pressure as usual. Here water displaces oil in the smallest available pores. But when water completely surrounds any region of oil, no further advance into that region will be possible, oil being incompressible. Such regions are called *trappings* and cause a problem generally known by the name of *residual oil*, an economic bane for oil industry.

In the above example the hydrophobic versus hydrophilic propertyplays an important role in the replacement of oil in pores with water. And water is prevented from penetrating trapped oil regions by much the same principle as that which prevents the water in the contents of a sandwich from crossing the spreaded layer of butter to wet the hydrophilic bread. For this latter case the pores in question are those within the bread texture, and the soaking of the bread is best prevented provided that the trapped regions of butter or margarine percolates in two dimensions to form a single layer or film which entirely covers the sectioning surface of the bread. Moreover, there is a similarity also in the internal structure of both the rock and the bread. The density of pores in bread is not homogeneous as a result of the tension on the surface of the dough caused by internal air pressure originated from the yeasts inside, as well as because of the heat applied to it when inside the oven, which dehydrolises the surface and makes it dry and hardened. The same inhomogeneity can be found inside the rocks which form the oil reservoirs where the regions of oil are surrounded by rocks which are less porous and have less permeability.

For Adler and Aharony (1988) a random walker, aka ant, treads on clusters. The ant enlarges a cluster by stepping on to an empty site next to it which meets certain conditions. They called this problem *diffusion percolation*. An example of a condition met is where the empty site has two or more occupied neighbours on a square lattice.

Most percolation in studies happens by randomly toggling the phases of sites or bonds in regular lattices. Kerstein (1983) considered randomly located spheres, take the complementary region of their union, and then perform percolation on the former. He showed that such percolation problem is equivalent to a percolation on a Voronoi lattice whose sites are the sphere centres.

In the same way as an infinite loop in computer science means that one can not come to the termination of a program going along the time dimension, an infinite cluster in percolation means that one can not come to the end of a cluster shifting along either one of the dimensions. The former case is along the one dimension of time and is only possible because of the time flows in one direction. Therefore half the line spanned is considered as being infinite. In one nondirectional dimension, except for the trivial case where one can consider the entire space as being one single cluster, no infinite cluster is possible. In simulation, when a cluster spans the whole of the space considered along any dimension we say that it is infinite since as one moves a cross section along that dimension, it always contains a section of the cluster.

The bond percolation program of two-dimensional Voronoi network by Tiyapan (1995, p. 78) takes $\mathrm{O}(n^2)$ time provided we assume that the contribution made by the number of clusters together with the that by each cluster comparison amount to a linear time complexity term. In order to justify this assumption, consider first the number of clusters. The maximum number of clusters possible depends on the size of the system, in other words it must vary as n_1^k, where $0 < k_1 < 1$. This maximum value should be approximately 0.5 because of symmetry between occupansy and void clusters. Next consider the time involved in comparing two clusters. On Matlab this is a sparse vector comparison which is likely to involves some linked lists, and similarly should take time as a function of n_2^k, $0 < k_2 < 1$. Because on average the size of clusters is always small before Pc, k_2 will be less than 0.5. Therefore $n_1^k \cdot n_2^k < n$ and it is safe to assume $\mathrm{O}(n)$ time from both of them combined. *q.e.d.*. A C translation (Tiyapan 1995, p. 80) of this program, though not as bad as it may seem because constant coefficients are small, gives $O(n^5)$ time in comparison.

In the field of geology Miller *et al* (2000) studies the analogy between the dilatant slip in

earthquakes and the hydrofraction occurred in melting and dehydration, the percolation of the latter in the permeability network internal to the fault being the cause of the former. According to them, the existence of toggle switches in the permeability rules out the assumption that the permeability throughout the whole system is homogeneous. Instead, the system reorganises itself into sytems of different scales of interaction according to the degree and nature of its inhomogeneity. At the critical state the scale of interaction is equal to the scale of the model.

The percolation probabilities given by Stauffer and Aharony (1994), the first number being for sites and the second for bonds, are for the honey comb lattice 0.6962, 0.65271; square 0.592746, 0.50000; triangular 0.50000, 0.34729; diamond 0.43, 0.388; simple cubic 0.3116, 0.2488; body-centred cubic 0.246, 0.1803; face-centred cubic 0.198, 0.119; 4-d hypercubic 0.197, 0.1601; 5-d hypercubic 0.141, 0.1182; 6-d hypercubic 0.107, 0.0942; 7-d hypercubic 0.089, 0.0787.

De Gennes *et al* (1959) investigated disordered binary solid solution AB where active atoms A are randomly replace the nodes of the periodic matrix B. There exists a critical concentration A in B below which all clusters are finite, and above which both finite and non-finite, *i.e.* infinite, clusters exist. Such solid solution in networks can represent the spin waves in alloys with one ferromagnetic component or the impurity bands in semiconductors. They cited seminal work on percolation by Broadbent *et al* (1957), but no mention was made about the Ising model.

In a way, percolation is similar to diffusion. In diffusion the particles considered move about randomly, whereas in percolation they can only crop up randomly at predetermined locations on a network which is fixed. We could imagine, for instance, cars running along the roads within a traffic network as diffusing through them. Then the percolation could occur on a larger scale, that is the scale of a road. The cars move along, that means they diffuse; but the roads remain fixed, and so their phases could percolate. In other words, in diffusion the particles move while in percolation, whether there are moving particles or not, it is the phases that percolate. Since historically percolation began as the study of diffusion of particles in a network of tubes in which the phases are naturally defined as the tubes being blocked or unblocked, these definitions have become most frequently used in other areas of application, for example in filtering membranes and traffic networks.

But this is not necessarily the case. Instead of dealing with a fixed network, one may consider a model of percolation in a continuum, for example by randomly patching an area until all the patches connect with one another somehow and percolate. The patch could be of any shape, as well as polygonal and circular. We can consider the percolation in a certain area as having occurred when there appears a cluster of patches which traverses any two opposite sides. One application of this is in the study of occurences of epidemics. Hoyle and Wickramasinghe (1979), having given a convincing argument in favour of viruses and various forms of diseases being carried to Earth from space by comets, talk about patchiness of pathogenic clouds. According to them, simultaneous attacks across vast region rules out person-to-person theory. Moreover, influenza epidemics are generally characterised by sudden onsets and equally sudden ends. These epidemics and plagues may be thought of as the percolation of these patches in a sufficiently large, predefined area. The bacteria and viruses coming from space adding to gene give the possibility of jump patterns in evolution (*cf* Smith and Szathmáry, 1995). The cells deliberately refuse to block viruses because they could prove to be useful in the long run, generally not by individuals but by the species. Historical examples are the disease described by Thucydides between 431 and 404 BC, five epidemics of 'English Sweats' between 1485 and 1552, and more than ten influenza pandemic from 1700 to 1900. Example of diseases which are caused by bacteria and viruses are bubonic plague, chicken pox (varicella), cholera, common cold, Legionnaires' disease, leprosy, measles, mumps, poliomyelitis, small pox, tuberculosis, and trachoma. Examples of major evolutionary transitions are those going from RNA to DNA, from prokaryotes to eukaryotes, and from axexual cloning to sexual propagation. Another example is the transition from primate to human both of whom differ from each other neurologically in the ability to use language and the power to conceptualise. One description of the Great Plague in London (Dickens, 1851). There was a rumour that a few people died in the winter of 1664. In May 1665 the disease burst out in St. Giles's which raged through July and September in every part of England; approximately 10,000 people died in London alone. But then the equinox winds virtually blew the disease away, and the Plague quickly disappeared. The existence of interstellar organic matters is supported by strong evidences with more and more complex substances constantly found (*cf* Hoyle and Wickramasinghe, 1978).

The growth mechanism of clusters in percolation is all so found in biology. Williams and Bjerknes (1972) simulate a tumour in the basal layer of an epithelium. The basal cells become less sensitive to Charlone which controls cellular division, and thus they divide κ times faster than the

normal cells where $\kappa > 1$ is the carcinogenic advantage. Abnormal cells interior to the basal layer divide, push, and then replace the neighbouring cells leaving the overall configuration unchanged except at the border where the abnormal cells exert a thrust of $\kappa - 1$ on their normal neighbours, that is $\frac{\mathrm{d}N}{\mathrm{d}t} = (\kappa - 1)n$ where n and N are respectively the numbers of peripheral and total abnormal cells. They found that dimensionality of fractal is involved and the dimension 1.1, instead of 1, must be assigned to the periphery, which means that n is proportional to $N^{0.55}$ not $N^{0.5}$. They found that abnormal cells push out faster in this order: the triangular, square, and hexagonal lattice. We may explain this, by looking at the coordination numbers of these three lattices, that the higher coordinaition number the lattice sites have, the slower the cluster expands. Added coordination means a higher degree-of-freedom the newly divided cells have to move about while still remaining local.

There is no percolation in one dimension because in such case the percolating cluster must necessarily contain the whole space. But there are some applications where the critical blockage is important, for example the blockage of drainage grilles by pea shingle is one-dimensional. Here the critical amount of blockage depends on the critical rate of flow which in turn depends on the amount of water and the rate of accumulation of water to be drained. The maintenance of gullies and grilles is done by cleaning, flushing and grit bucket emptying (*cf* Harrison and Trotman, 2002).

§ 3.1 Statistics of continuum percolation in literature

Mecke and Seyfried (2002) find pseudocritical threshold for different types of lattices and then extrapolate these by using finite-size scaling laws to obtain the critical probability for the infinite network. They use the Hoshen-Kopelman method to find the largest cluster of the system and then determine the percolation in fifteen steps by increasing or decreasing p by $\frac{\Delta p}{2}$ depending on whether the system percolates or not. Here $\Delta p = p_2 - p_1$, $p_2 > p_1$, such that the system only percolates at p_2 not p_1. In two dimensions the network percolates when two opposite sides connect with each other whereas in three dimensions the same is true when four opposite sides connect together. The pseudocritical percolation threshold is $p_c(L)$, where L is the network size. The finite-size scaling theorem is $p_c(L) = p_c(\infty) + aL^b$. In two dimensions they find p_c =0.59278(4), while in three dimensions $p_c = 0.31162(8)$.

Tiyapan (1995) finds from 27 simulations on 2-d Voronoi networks of between 70 and 500 cells the percolation probability of cells $p_c = 0.507$, and from 75 simulations on 2-d Voronoi networks of between 100 and 8,600 bonds p_c =0.658. For honey comb lattices between 100 and 3,000 bonds he finds the bond percolation probability $p_c = 0.640$, for Kagome lattices between 300 and 3,000 bonds $p_c = 0.517$, for square lattices between 200 and 2,000 bonds $p_c = 0.467$, and for triangular lattices between 300 and 3,000 bonds $p_c = 0.341$. These results from simulations are usually less than the exact values by a few per cent.

§ 3.2 Continuum percolation

Consider the continuum percolation of regular polygons in two dimensions. Polygons are placed on the plane randomly with regards to both their position and orientation. From their position one can find the Delaunay triangulation. Using the bonds of the triangulation obtained as reference axes, one for each pair of polygons, one then finds the orientation of any two neighbouring polygons with reference with the axis connecting them.

Unlike in the case of Voronoi percolation where the size of each cell depends on the density of the generating points, in continuum percolation the size of the polygons has been decided in advance. The expanse of the region of interest is then determined relative to this size. The area being simulated can then be imagined as lying within an infinite plane where the number density of polygons is homogeneous.

The area of the n-gon is $A = n \cdot 2 \cdot \frac{1}{2} yz$. But $y = R \sin\theta$ and $z = R\cos\theta$, where R is the radius of the circumscribing circle. Therefore we have $R = \frac{A}{n \sin\theta \cos\theta}$. The radius of the inscribed circle is $r = z$.

A list is made of every angle that the rays perpendicular to the sides of the polygons make with the x-axis in the counterclockwise direction. Then the key procedure of the algorithm is to find for each pair the ray which lies closest to the reference line joining the two polygons. The angles that these two rays make with the reference axis determines whether or not the two polygons intersect.

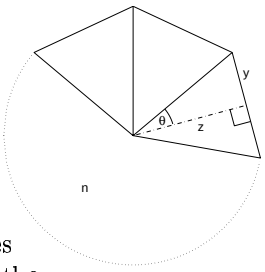


Figure 8

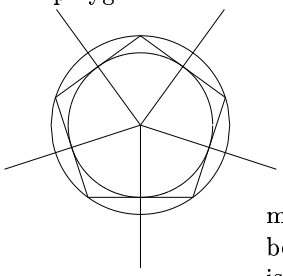


Figure 3.2

The inscribed circle and the circumscribing circle both play a role in deciding whether two polygons form a cluster. Let the radius of the former be represented by r and the radius of the latter R. If any two polygons are less than $2r$ apart, then they must overlap. They just touch if they are exactly $2r$ apart. And if the distance between them is more than $2R$, then they can never touch each other. When the distance lies between $2r$ and $2R$ there is a probability that they will touch. Whether this is the case or not depends on the orientation of the two polygons.

Among all the rays similar to those shone in Figure 3.3 one can find the one closest to the line between the centres of the two polygons, ie θ_1 and θ_2 of Figure 3.4 are a pair of minimum angles for c_1 and c_2. Let the centres of these polygons be c_1 and c_2, and the line joining them $\overline{c_1c_2}$. Two such lines obtained from the two polygons can fall into on of the two cases; they can either be on the same side of $\overline{c_1c_2}$, or they are on the opposite side of each other as shown in Figure 3.4 (a) and (b) respectively. In either case, the following theorem is true.

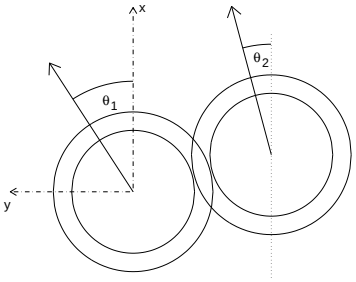


Figure 3.3

Theorem 3.1. *The line segments $\overline{a_1a_2}$ and $\overline{b_1b_2}$ always intersect $\overline{c_1c_2}$.*

Proof.: Without losing generality, consider only the case of $\overline{a_1a_2}$. Suppose the above statement is false and $\overline{a_1a_2}$ did not intersect $\overline{c_1c_2}$. Then, because $\overline{a_1a_2}$ is a part of the circumference of a polygon, there would always be another ray d_k originating from c_1 which has $\overline{a_ia_j}$ connected to it, such that $\overline{a_ia_j}$ intersects $\overline{c_1c_2}$, i, j and k being positive intergers. But the two triangles $\triangle c_1a_1a_2$ and $\triangle c_1a_ia_j$ are identical. Therefore the angle θ_k between $\overline{a_ia_j}$ and $\overline{c_1c_2}$ would be smaller than θ_1 and we replace d_1 with d_k because the latter is closer to $\overline{c_1c_2}$. □

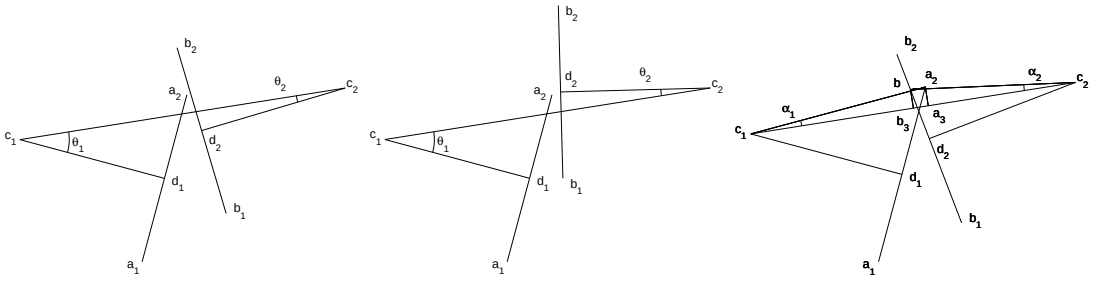
Furthermore, between θ_1 and θ_2 the one which is greater would belong to an *aggressive* cell, while the other one is in a *passive* state. Suppose that $\theta_1 > \theta_2$, then c_1 would be the one which does the touching first. Theorem 3.2 states this in a more formal manner.

Theorem 3.2. *If the distance between any two identical regular polygons allows them a certain probability p of overlap, such that $0 \leq p \leq 1$, then the polygon whose ray is furthest from the line connecting their centres will be the one which causes the two to intersect.*

Proof.: Let c_1 and c_2 be the two identical polygons in question. As mentioned earlier, there are two cases to be considered, which are represented by Figure 3.4 (a) and (b). Because the distance

between them is more than the diameter of the inscribed circle, when $\theta_1 = \theta_2 = 0$ the two polygons do not touch each other. The two polygons being identical, it is suffice to consider only one of them. But because there is an overlap area between the two as shown in Figure 3.3 where there is a probability of colliding to occur, and as the distance from a centre c_1 increases as one goes from a midpoint d_1 of an edge to the next vertex d_2 closest to $\overline{c_1c_2}$, the greater θ_1 is, the further away from c_1 is the point of intersection between $\overline{d_1b_2}$ and $\overline{c_1c_2}$, the more that polygon overlaps into the collision hazard zone mentioned above, and therefore the more contribution to the probability of the collision. □

Alternative proof. As shown in Figure 3.4 (c), with $\theta_1 > \theta_2$ it is the corner a_2 which collides with the flat of the side b_1b_2 and penetrates into the second polygon. Imagine the polygons revolving around until they collide. At the point of collision if their vertices touch, then $\theta_1 = \theta_2$ and it is a special case where both contribute equally to the touch. The case of Figure 3.4 (b) where the two angles lie on the opposite side of each other follows a similar line of reasoning. □



(a) (b) (c)

Figure 3.4 *The distance between c_1 and c_2 is less than the diameter of the circumscribing circle but more than the diameter of the inscribed circle.*

If $\theta_1 > \theta_2$, then $d(\overline{c_1a_3}) > d(\overline{c_2b_3})$. Furthermore, a touch or penetration implies that $d(\overline{c_1a_3}) + d(\overline{c_2b}) = d(\overline{c_1c_2})$. If $d(\overline{c_1a_3}) + d(\overline{c_2b}) < d(\overline{c_1c_2})$ then the two polygons do not touch each other. The case when they merely touch at vertices is the case where $d(\overline{c_2b}) = d(\overline{c_2b_3})$. On the other hand, the condition $d(\overline{c_1a_3}) + d(\overline{c_2b_3}) \geq d(\overline{c_1c_2})$ does not say anything much because a_2 may just pass under b_2 without touching the side in Figure 3.4 (a), or it may be the case where the two opposing sides lean towards the same direction as in Figure 3.4 (b). Therefore, we now have the following theorem which is later used in Algorithm 3.1 for finding overlaps.

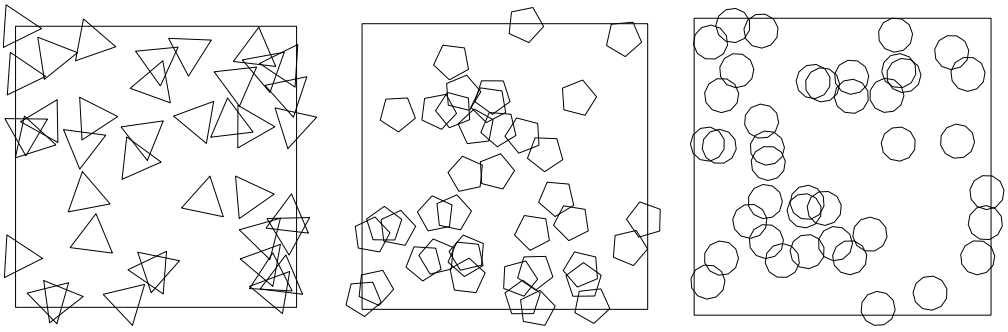
Theorem 3.3. *Any two polygons p_1 and p_2 touch each other if and only if $d(\overline{c_1a_3}) + d(\overline{c_2b}) = d(\overline{c_1c_2})$.*

Proof.: See explanation above together with Figure 3.4. □

Algorithm 3.1 Find continuum percolation of regular polygons on a plane.

place polygons randomly;
find radii of their inscribed circles and the circumscribing circles;
find coordinates of the vertices;
find Delaunay triangulation;
for $i = 1$ **to** *number of triangles* **do**
  for $j = 1$ **to** 3 **do**
    find *distance between the j^{th}-pair vertices of the i^{th} triangle;*
    if this *distance* $\leq 2r$ **then** polygons overlap **do**
      record cells overlap;
    elseif *distance* $\leq 2R$ **then** polygons could overlap **do**
      record cells to find whether overlap;
    endif
  endfor
  find *angle* of each of the two polygons when viewed from the other
endfor
for *every side of every polygon* **do**
  find *angles of the normal ray counterclockwise from the positive x-axis;*
endfor
for *every cell which could still overlap* **do**

find *normal rays closest to the lines going towards its neighbours;*
endfor
complete the overlap check for all remaining cells;
check percolation;



(a) (b) (c)

Figure 3.5 **(a)** Randomly placed *triangles.* Here the program tells us that the largest cluster has seven triangles, and the next largest one has six. There are 8 isolated triangles, 5 clusters of two, and three clusters of three. According to the program, the two clusters of sizes six and two on the upper right corner do not quite touch. **(b)** The largest cluster in this case has got nine *pentagons.* **(c)** These 11-*gons* are also too sparsely placed to percolate. The largest cluster in this case has got 8 members. The number of all 11-gons is 40, the same as in the two previous cases, and six of which are isolated. There are one cluster of four, four clusters of three, and five clusters of two.

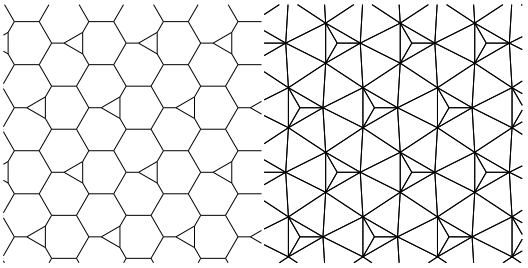
A percolation threshold normally means the critical percentage by number.

§ 3.3 Tilings and patterns

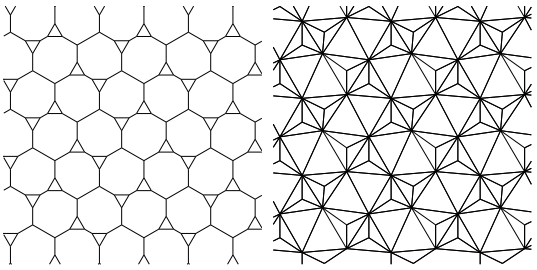
All two dimensional lattices are tilings. The *kagome* latticedeserves some mention here since it is a first-order covering lattice of the honeycomb lattice, which in turn is dual to the triangular lattice. Some physicists conjecture that the name originates from a name of a person. Apart from a few exceptions from the reason that the word is written in the wrong script, those scientists who are japanese should know that the word from their own language which means, according to the dictionary by Nelson (1962), *basket interstices* or *woven bamboo pattern.* My own version of the translation is simply *basket pattern* (*cf* Tiyapan, 2001), *kago* (*rad take* and *tatsu*) meaning *basket* and *me* a radical word for *eye* which also means *pattern.* Many researchers and students, blindly following an established practice, often spell it as *kagomé* (*cf*, for example, Tiyapan, 1995) with an acute accent or *l'accent aigue* like a french word.

§ 3.4 2-homohedral tilings

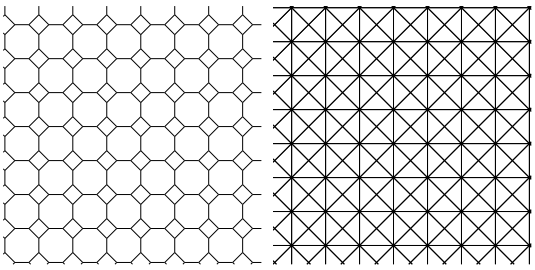
There are altogether 39 types of 2-homohedral tilings, ie. those which have vertices of the same valence. Of these, there are 26 types of valence 3, 10 of valence 4, and 3 of valence 5 (*cf* Grünbaum *et al*). The code and the data used in the simulations are in § 7.6. In the program, the variable o contains the vertex numbers which must be ordered by scanning the basic tile from left to right and gradually from bottom up. Unless the variable follows this ordering strange results will occur. The variables m and n are respectively the x- and y-coordinates of the vertices.



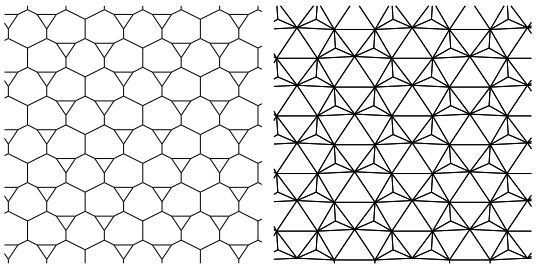
(1) $3_3[3^3]7_1[3^7]$



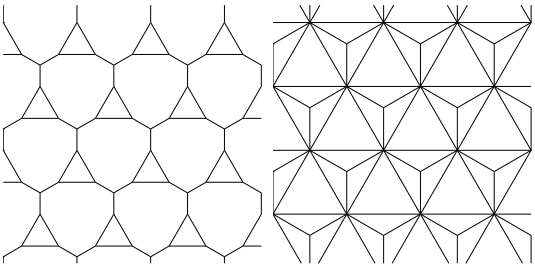
(2) $3_3[3^3]9_3[3^9]_{\mathrm{II}}$



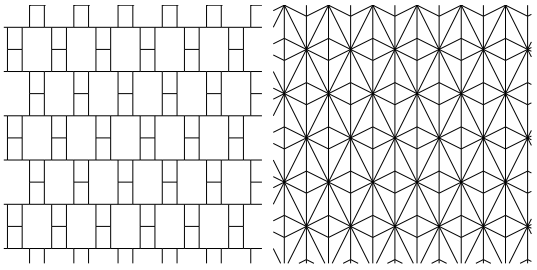
(3) $4_4[3^4]8_4[3^8]$



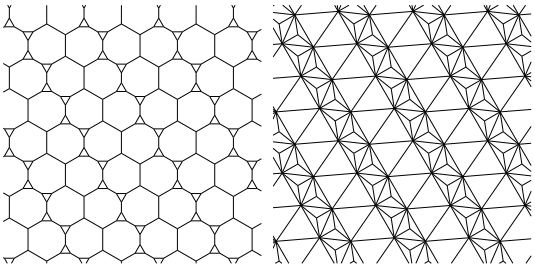
(4) $3_3[3^3]8_2[3^8]$



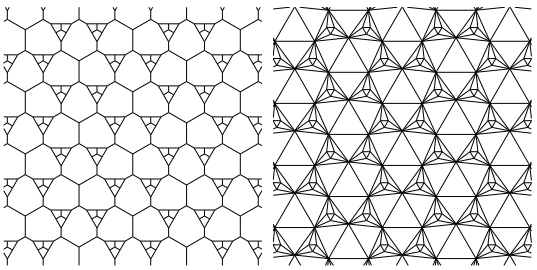
(5) $3_3[3^3]9_3[3^9]_{\mathrm{III}}$



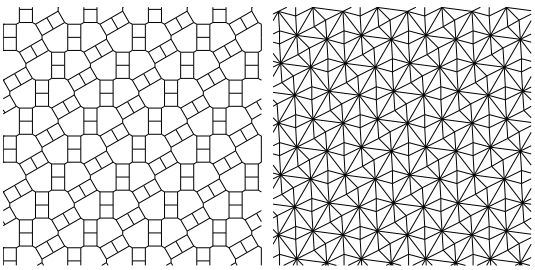
(6) $4_3[3^4]10_6[3^{10}]_{\mathrm{I}}$



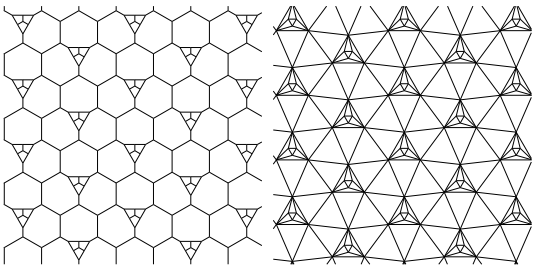
(7) $3_3[3^3]9_3[3^9]_{\mathrm{I}}$



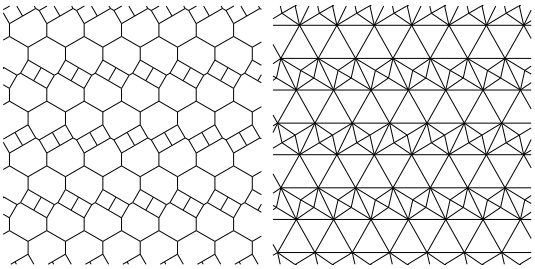
(8) $4_2[3^4]10_4[3^{10}]$



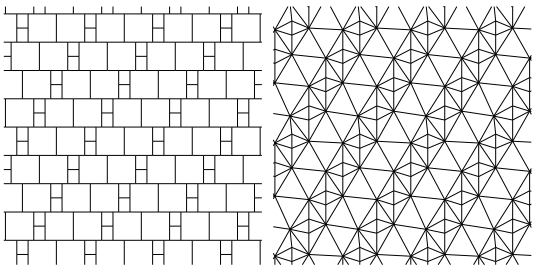
(9) $4_3[3^4]10_6[3^{10}]_{\mathrm{II}}$



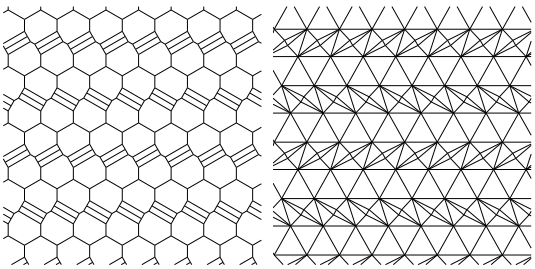
(10) $4_2[3^4]8_2[3^8]$



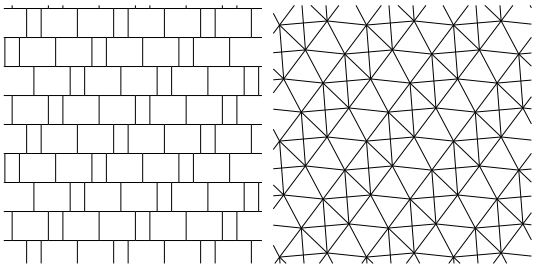
(11) $4_3[3^4]8_3[3^8]_{\mathrm{I}}$



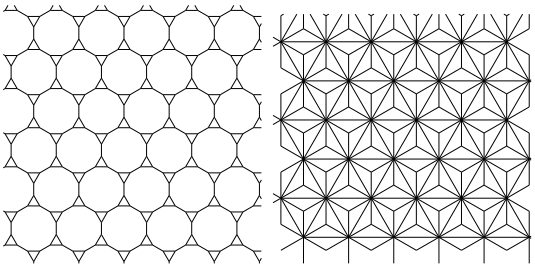
(12) $4_3[3^4]8_3[3^8]_{\mathrm{II}}$



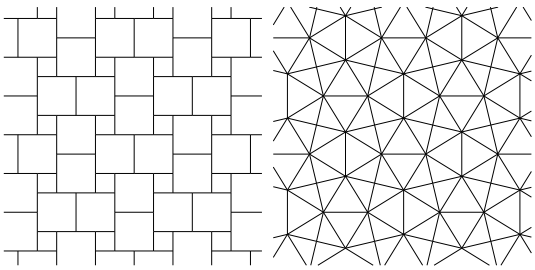
(13) $4_3[3^4]8_3[3^8]_{\mathrm{III}}$



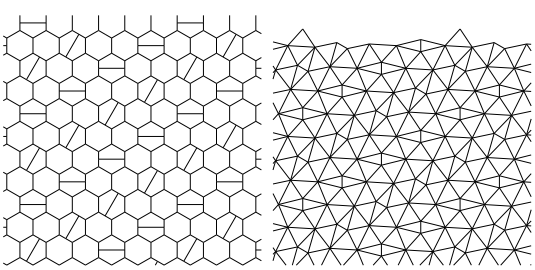
(14) $4_4[3^4]7_2[3^7]_{\mathrm{II}}$



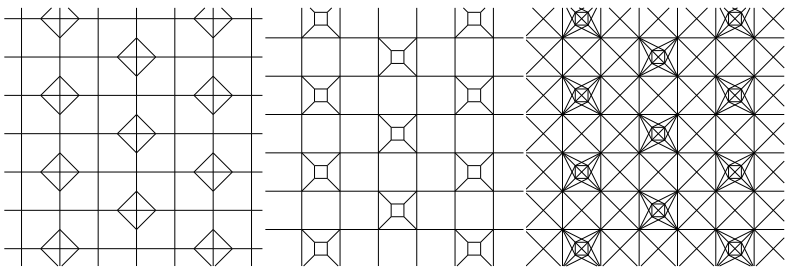
(15) $3_3[3^3]12_6[3^{12}]$



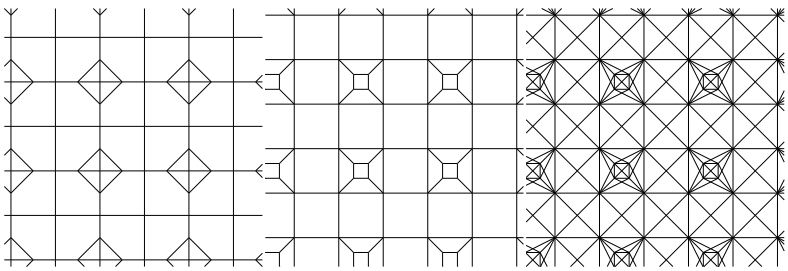
(16) $4_4[3^4]7_2[3^7]_{\mathrm{I}}$



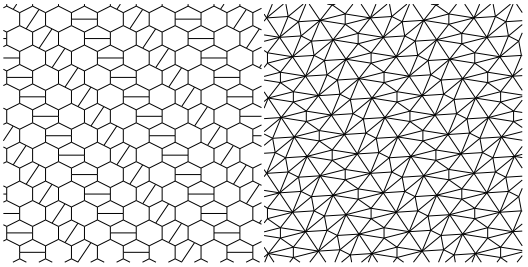
(17) $5_4[3^5]7_4[3^7]_{\mathrm{I}}$



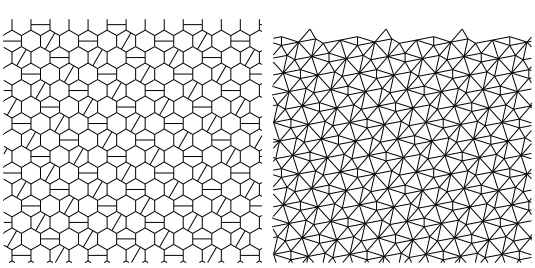
(18) $3_1[4^3]5_1[4^5]_{\mathrm{I}}$



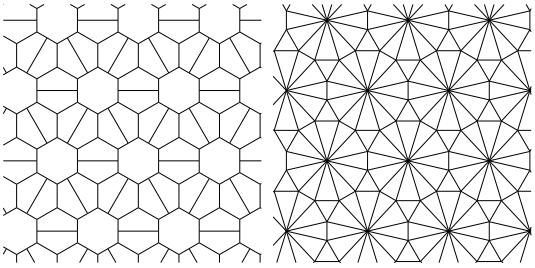
(19) $3_1[4^3]5_1[4^5]_{\mathrm{II}}$



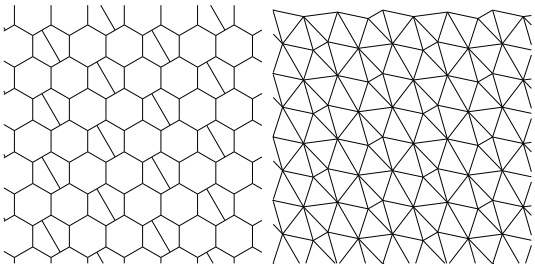
(20) $5_3[3^5]8_6[3^8]_{\mathrm{II}}$



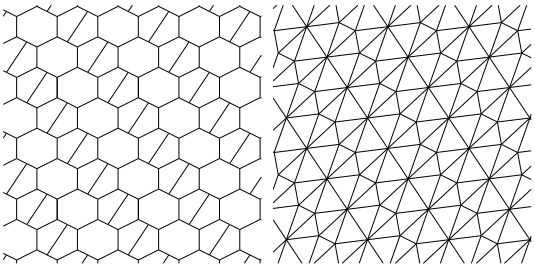
(21) $5_3[3^5]8_6[3^8]_{\mathrm{III}}$



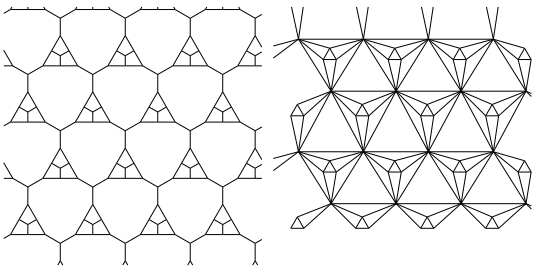
(22) $5_2[3^5]12_{12}[3^{12}]$



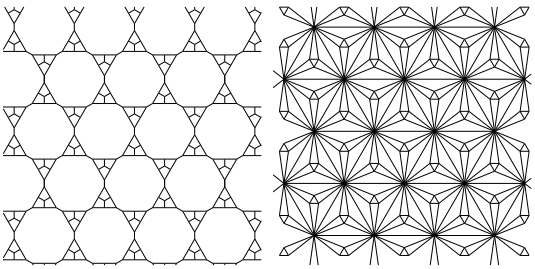
(23) $5_4[3^5]7_4[3^7]_{\mathrm{II}}$



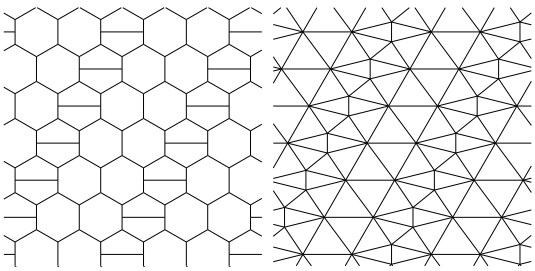
(24) $5_3[3^5]8_6[3^8]_{\mathrm{I}}$



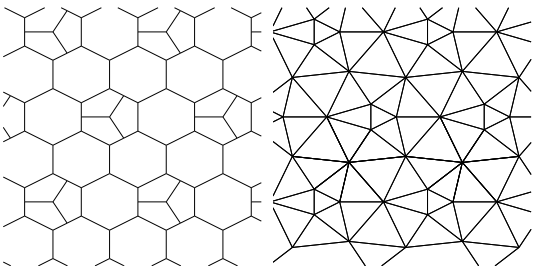
(25) $4_2[3^4]12_6[3^{12}]$



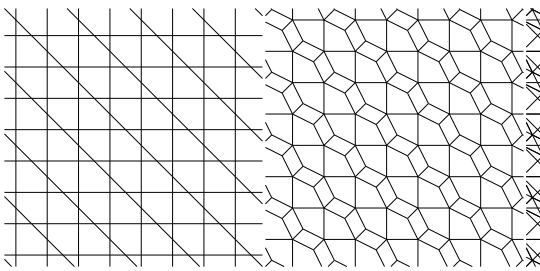
(26) $4_2[3^4]18_{12}[3^{18}]$



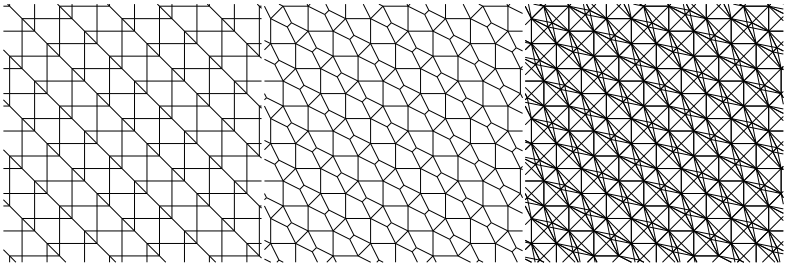
(27) $5_3[3^5]7_3[3^7]_{\mathrm{I}}$



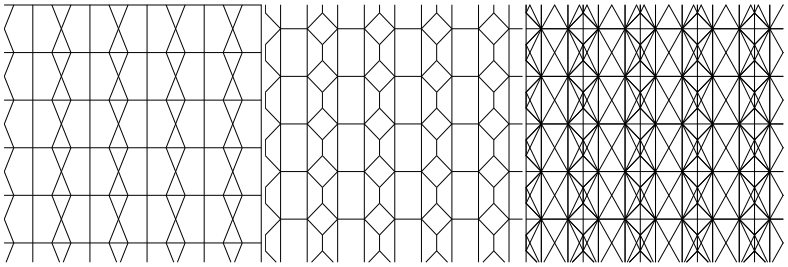
(28) $5_3[3^5]7_3[3^7]_{\mathrm{II}}$



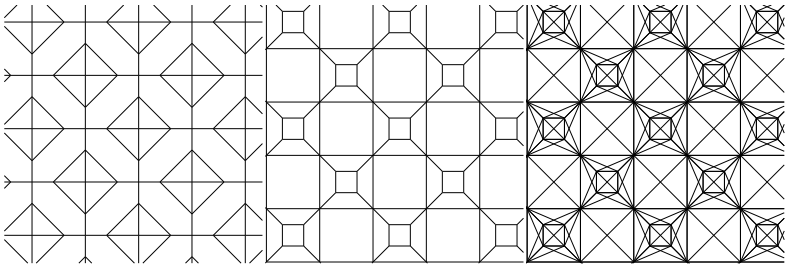
(29) $3_2[4^3]5_2[4^5]_{\mathrm{I}}$



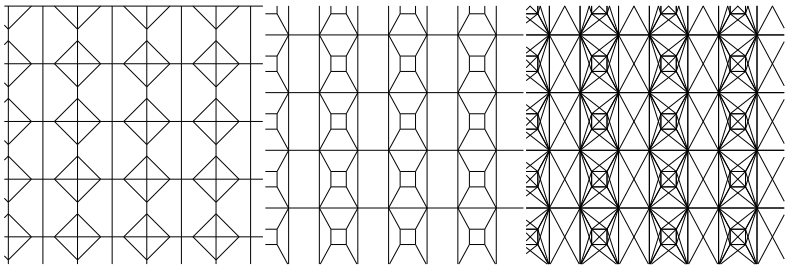
(30) $3_2[4^3]5_2[4^5]_{\mathrm{II}}$



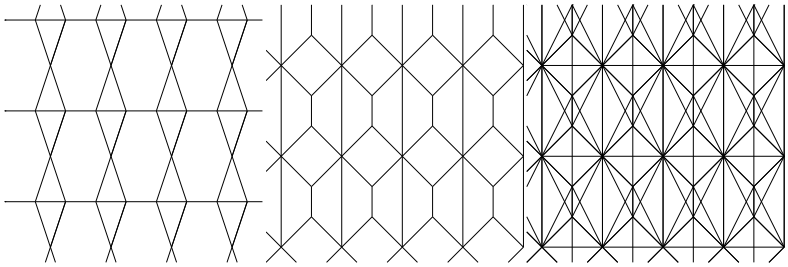
(31) $3_3[4^3]5_3[4^5]$



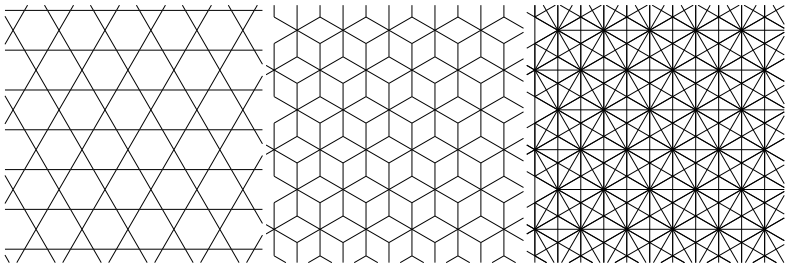
(32) $3_1[4^3]6_2[4^6]_{\mathrm{I}}$



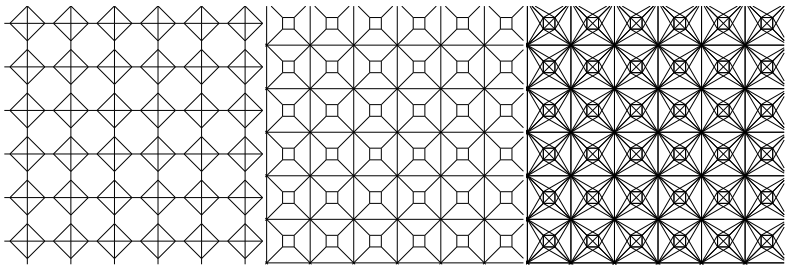
(33) $3_1[4^3]6_2[4^6]_{\mathrm{II}}$



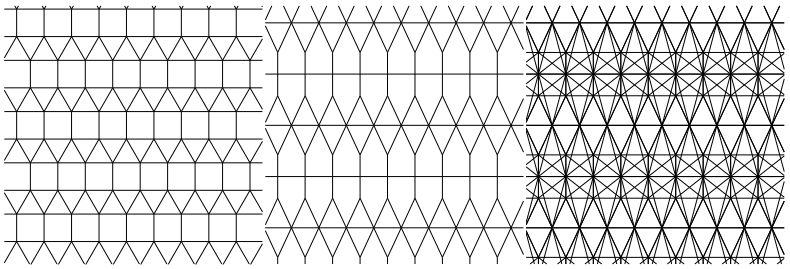
(34) $3_2[4^3]6_4[4^6]$



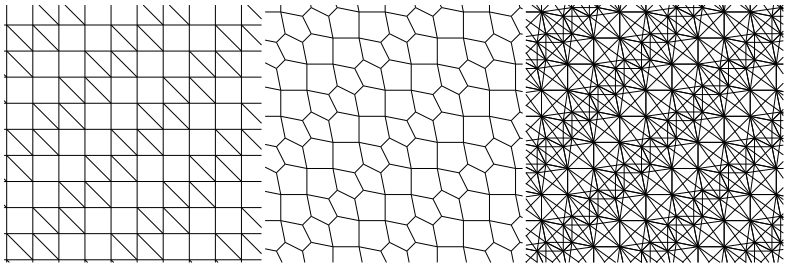
(35) $3_3[4^3]6_6[4^6]$



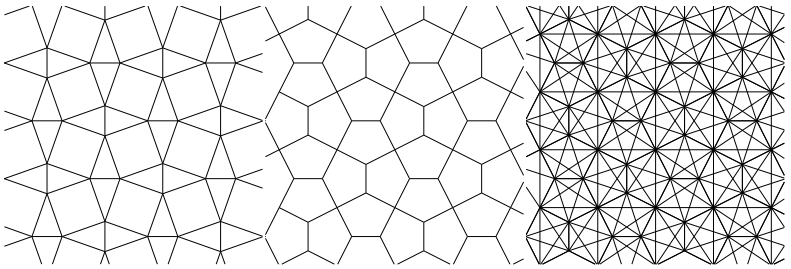
(36) $3_1[4^3]8_4[4^8]$



(37) $3_1[5^3]4_2[5^4]_{\mathrm{I}}$



(38) $3_1[5^3]4_2[5^4]$



(39) $3_2[5^3]4_4[5^4]$

Figure 3.1 *The 39 2-homohedral tilings.*

Once while doing the above simulations I came across an interesting batch of the $4_4[3^4]7_2[3^7]_{\mathrm{II}}$ 2-homohedral tiling where the percolations forward and backward in time gave the same value of p_c, that is there is an approximate time symmetry. In other words the p_c's of both the first and the second population are symmetric to each other. Each percolates when its population reaches 756 out of the combined total of 1070, which identically gives $p_c = 0.7065$. Large non-percolating clusters quickly disappears after the onset of percolation as shown in Figure 3.2. In Figure 3.2 (b) the size of the second largest cluster is reduced to one between the population II of 941 and 961, which correspond to the density of 0.8794 and 0.8981 respectively. Interestingly from the population of 961 until 1070 there is only one cluster despite the fact that there are still more than one hundred population I. Here *population II* always refers to any population which percolates or which is being observed with regard to percolation. On the other hand in Figure 3.2 (b) the second biggest cluster is the smallest cluster from population 898 to 1037. Again from the population of 1038 until the maximum 1070 no second clusters appear.

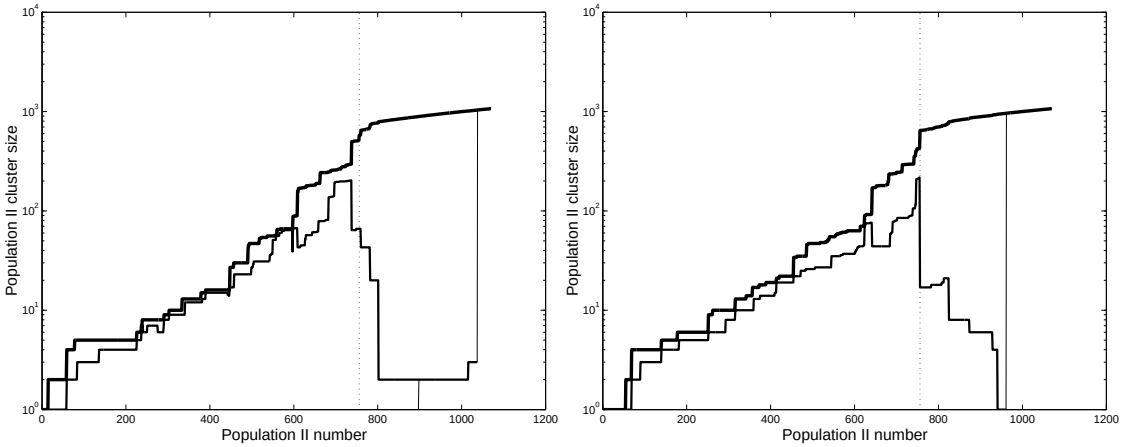
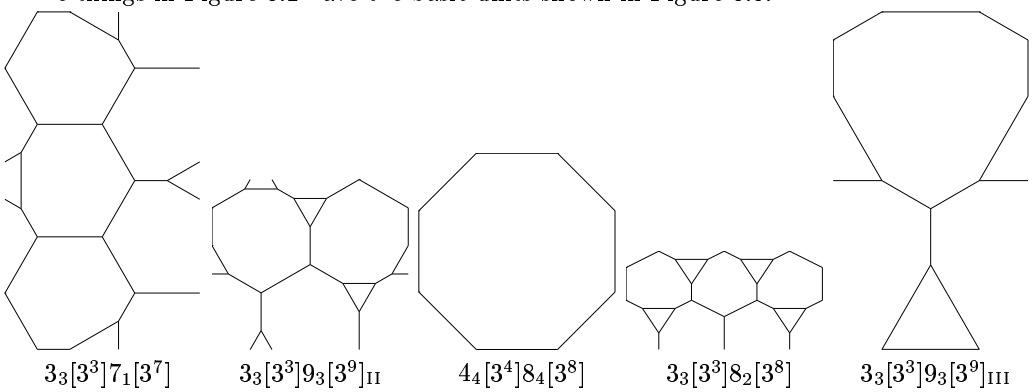
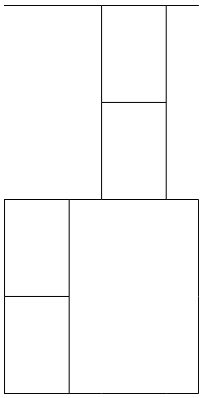


Figure 3.2 *The biggest, the second biggest, and the smallest cluster from a batch of the* $4_4[3^4]7_2[3^7]_{\mathrm{II}}$ *2-homohedral tiling where the percolation happens to be symmetry in time, that is (a) and (b) are time symmetric to each other. The vertical dotted line is where the percolation point occurs in each case. The the heaviest to the lightest lines are respectively the progression of the clusters which are biggest, second biggest, and smallest at each population density.*

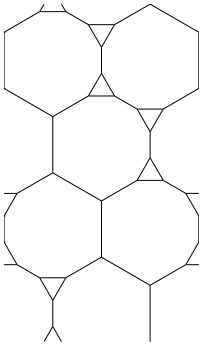
The percolation considered is that which percolates from the lower bound to the upper bound, both of which are fixed. Because the networks must be constructed such that there would be duplicates of neither vertices nor links, there must be cells some vertices and edges of which belong to one or more of the basic tiles adjacent to them. As a result, those basic tiles on the boundary sometimes have a few of their cells missing, and depending on whether the basic tile in question is very complex or simple the position of the missing cell may merely lie on its boundary or may lie nearly halfway into its interior as is the case with the $4_3[3^4]8_3[3^8]_{\mathrm{I}}$ and the $4_3[3^4]8_3[3^8]_{\mathrm{III}}$ tilings. The upper and the lower boundaries, which decide the percolation, are chosen as straight lines parallel to the vertical axis and lying in the direction towards the interior of the network and at a fixed distance from the the maximum and the minimum horizontal positions respectively of all the vertices in the network. This fixed distance is defined for each network to be a fixed ratio of its overall width. The value of 0.05 had been used which was later changed to 0.1 due to a problem which occured while simulating the $4_2[3^4]8_2[3^8]$ tessellation, missing cells formed a continuous band disconnecting all the cells on the boundary from the rest of the network, the upper boundary lay within this band when shifted by five percent from the upper limit but not when shifted by ten percent.

The tilings in Figure 3.2 have the basic units shown in Figure 3.3.

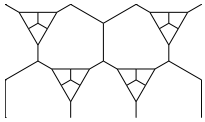




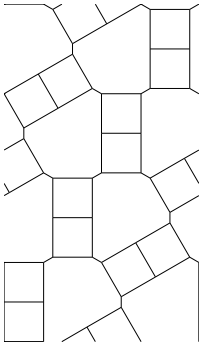
$4_3[3^4]10_6[3^{10}]_{\mathrm{I}}$



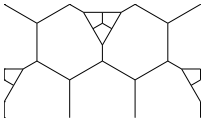
$3_3[3^3]9_3[3^9]_{\mathrm{I}}$



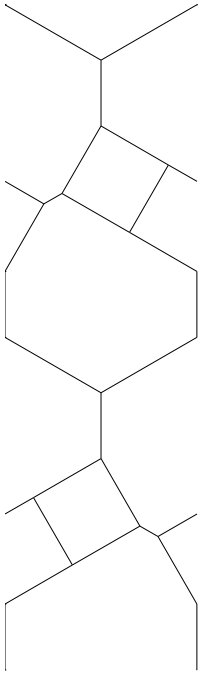
$4_2[3^4]10_4[3^{10}]$



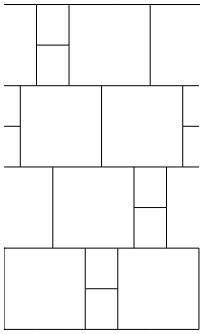
$4_3[3^4]10_6[3^{10}]_{\mathrm{II}}$



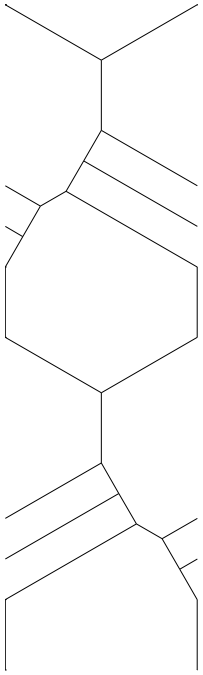
$4_2[3^4]8_2[3^8]$



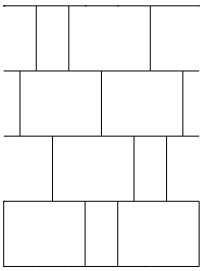
$4_3[3^4]8_3[3^8]_{\mathrm{I}}$



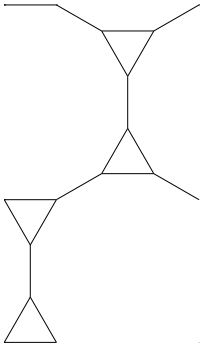
$4_3[3^4]8_3[3^8]_{\mathrm{II}}$



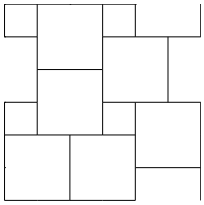
$4_3[3^4]8_3[3^8]_{\mathrm{III}}$



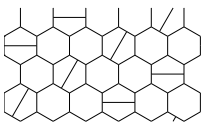
$4_4[3^4]7_2[3^7]_{\mathrm{II}}$



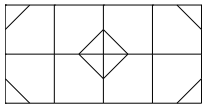
$3_3[3^3]12_6[3^{12}]$



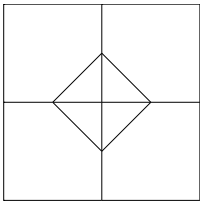
$4_4[3^4]7_2[3^7]_{\mathrm{I}}$



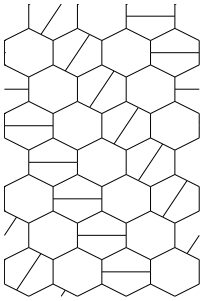
$5_4[3^5]7_4[3^7]_{\mathrm{I}}$



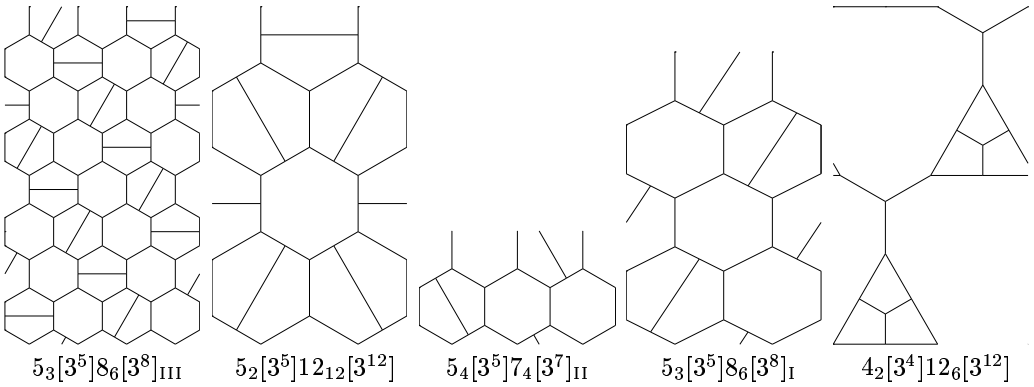
$3_1[4^3]5_1[4^5]_{\mathrm{I}}$



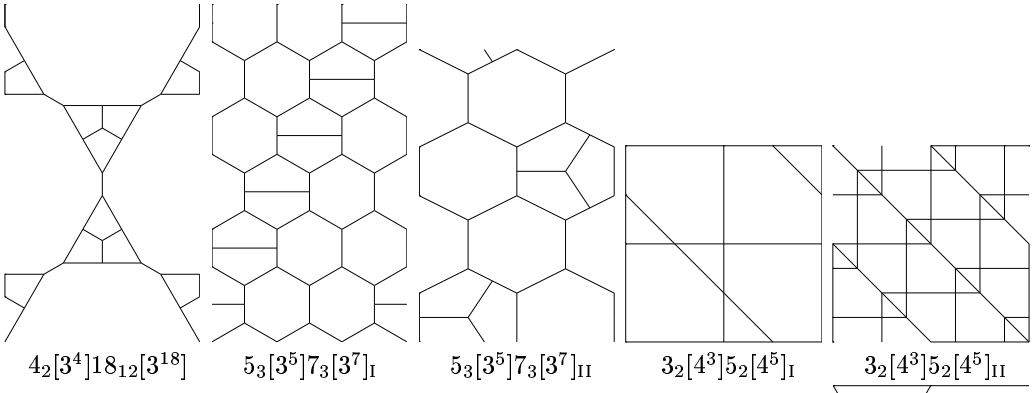
$3_1[4^3]5_1[4^5]_{\mathrm{II}}$



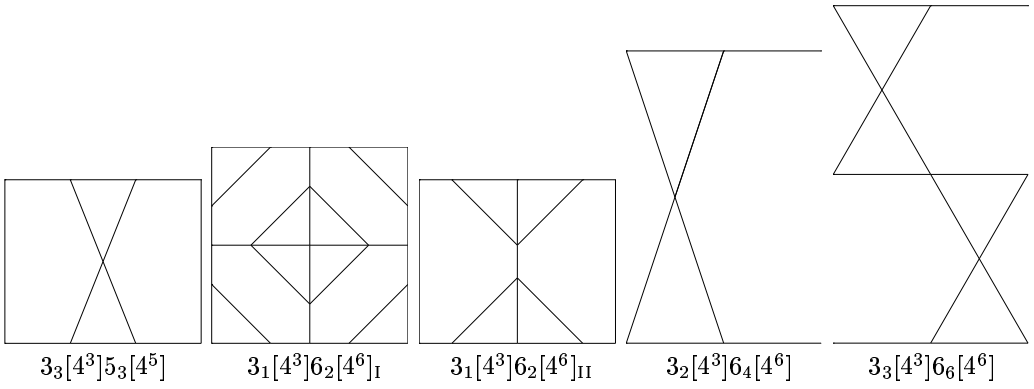
$5_3[3^5]8_6[3^8]_{\mathrm{II}}$



$5_3[3^5]8_6[3^8]_{III}$ $5_2[3^5]12_{12}[3^{12}]$ $5_4[3^5]7_4[3^7]_{II}$ $5_3[3^5]8_6[3^8]_I$ $4_2[3^4]12_6[3^{12}]$



$4_2[3^4]18_{12}[3^{18}]$ $5_3[3^5]7_3[3^7]_I$ $5_3[3^5]7_3[3^7]_{II}$ $3_2[4^3]5_2[4^5]_I$ $3_2[4^3]5_2[4^5]_{II}$



$3_3[4^3]5_3[4^5]$ $3_1[4^3]6_2[4^6]_I$ $3_1[4^3]6_2[4^6]_{II}$ $3_2[4^3]6_4[4^6]$ $3_3[4^3]6_6[4^6]$

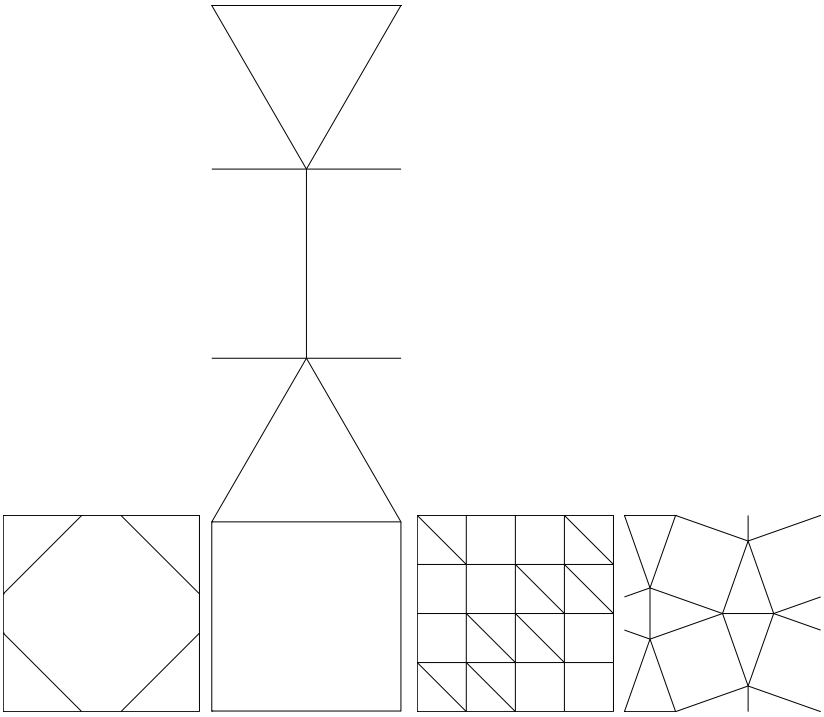


Figure 3.3 *The unit cells of the* 2*-homohedral tilings.*

In Table 3.1 there are six types of percolation probabilities. Two cells can be neighbours to each other under the criterion of having either one or two vertices in common. This gives rise to two different values of p_c's for cells, and another two for bonds. These can be termed 1-neighbours and 2-neighbours respectively. In Physics and other physical sciences 2-neighbours are more important because the greater the contact area is the stronger the bonding. This is not always the case in other fields, for example in Sociology a smaller point of contact sometimes implies the less duplications of neighbours' neighbours, and thus the larger the resultant networks. Because there is a cost for maintaining the connections, the smaller each contact point is and the larger the network the better, therefore here 1-neighbours seem to be more important.

	Cell	Bond	cell	bond	vertex	edge
$3_3[3^3]7_1[3^7]$	0.4714 ± 0.0667 1.1551×10^{-6} 4.0885×10^{-5}	0.3041 ± 0.0272 -3.3740×10^{-6} 1.1316×10^{-6}			0.7212 ± 0.0444 -1.1124×10^{-4} 1.9314×10^{-5}	0.6771 ± 0.0381 -5.2007×10^{-5} 9.5566×10^{-6}
	5[99], 4[195], 5.2525 5.4667 12[391] 5.6215	12[1099] 10.4149			4[256], 4[470], 2.7266 2.7957 12[894] 2.8523	4[349], 4[657], 3.7077 3.7869 12[1275] 3.8494
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$3_3[3^3]9_3[3^9]_{\rm II}$	0.4966 ± 0.0655 -8.2288×10^{-5} 3.7765×10^{-5}	0.2734 ± 0.0206 -5.5513×10^{-6} 5.9281×10^{-7}	0.5042 ± 0.0574 -1.0585×10^{-4} 2.1108×10^{-5}	0.3119 ± 0.0417 4.8211×10^{-5} 7.5982×10^{-6}	0.7628 ± 0.0228 1.0042×10^{-6} 7.3299×10^{-7}	0.7094 ± 0.0435 -2.7590×10^{-5} 7.7345×10^{-6}
	4[176], 2[139], 5.2727 5.1942 2[305], 6[412] 5.4492 5.5243	6[831], 6[1138] 12.0144 12.1564	2[305], 6[412] 5.0557 5.1165	2[771], 6[1054] 11.0298 11.1556	4[415], 2[334], 2.8193 2.7964 2[692], 6[919] 2.8584 2.8770	4[585], 2[467], 3.8120 3.7859 2[989], 6[1322] 3.8544 3.8744
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$4_4[3^4]8_4[3^8]$	0.4695 ± 0.0858 -3.5574×10^{-5} 1.4884×10^{-4}	0.3044 ± 0.0429 2.8967×10^{-5} 6.8367×10^{-6}			0.7194 ± 0.0209 -5.8481×10^{-6} 6.5739×10^{-7}	0.6407 ± 0.0296 2.8990×10^{-6} 2.0123×10^{-6}
	2[181], 4[265], 5.5691 5.6453 12[313] 5.6741	4[748], 12[888] 10.5294 10.5946			1[440], 2[624], 2.7273 2.7692 6[728] 2.7857	2[600], 2[864], 3.7333 3.7778 6[1014] 3.7949
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$3_3[3^3]8_2[3^8]$	0.4903 ± 0.1015 1.2205×10^{-4} 2.1262×10^{-4}	0.2926 ± 0.0281 -1.8541×10^{-5} 1.9682×10^{-6}			0.7447 ± 0.0329 2.1925×10^{-7} 2.8427×10^{-6}	0.7019 ± 0.0270 -6.4679×10^{-6} 8.9933×10^{-7}
	2[166], 4[301], 5.1928 5.3953 12[476] 5.5168	4[812], 12[1313] 11.0788 11.2749			2[381], 2[668], 2.8346 2.8743 6[1035] 2.8986	2[540], 2[960], 3.8444 3.8833 6[1500] 3.9067

	Cell	Bond	cell	bond	vertex	edge
$3_3[3^3]9_3[3^9]_{III}$	0.4607 ± 0.0586 7.9409×10^{-5} 3.3840×10^{-5}	0.2483 ± 0.0258 4.8742×10^{-6} 7.4553×10^{-7}			0.7482 ± 0.0264 2.4849×10^{-6} 1.2161×10^{-6}	0.7049 ± 0.0195 -1.0461×10^{-5} 1.0354×10^{-6}
	2[83], 4[357], 4.6988 5.3613 12[438] 5.4247	4[957], 12[1188] 12.0878 12.1785			2[212], 2[808], 2.6792 2.8317 6[980] 2.8469	2[284], 2[1144], 3.6620 3.8322 6[1395] 3.8480
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$4_3[3^4]10_6[3^{10}]_I$	0.5286 ± 0.0511 5.4350×10^{-5} 1.7402×10^{-5}	0.2991 ± 0.0319 -8.0790×10^{-6} 2.7687×10^{-6}			0.7789 ± 0.0386 8.7871×10^{-6} 4.7653×10^{-6}	0.7173 ± 0.0391 -1.2114×10^{-4} 1.8197×10^{-5}
	8[188], 4[295], 5.3085 5.4441 12[426] 5.5352	8[499], 4[803], 11.2826 11.5691 12[1179] 11.7574			4[429], 2[656], 2.6853 2.7439 6[931] 2.7841	4[576], 2[900], 3.8333 3.8667 6[1296] 3.8889
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$3_3[3^3]9_3[3^9]_I$	0.5206 ± 0.0551 -9.0646×10^{-6} 1.7837×10^{-5}	0.2987 ± 0.0308 7.4580×10^{-6} 1.7780×10^{-6}			0.7532 ± 0.0296 -8.9673×10^{-6} 1.3152×10^{-6}	0.7063 ± 0.0222 -4.5643×10^{-6} 7.3255×10^{-7}
	4[259], 4[473], 5.3977 5.5518 6[751] 5.6431	4[699], 4[1313], 11.9371 12.2224 6[2119] 12.3870			2[594], 2[1048], 2.8485 2.8855 6[1630] 2.9080	2[846], 2[1512], 3.8440 3.8836 6[2370] 3.9072
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$4_2[3^4]10_4[3^{10}]$	0.4726 ± 0.0684 1.0814×10^{-4} 8.0038×10^{-5}	0.2905 ± 0.0633 5.5163×10^{-4} 1.3835×10^{-4}			0.7341 ± 0.0316 -8.0603×10^{-6} 2.0508×10^{-6}	0.6957 ± 0.0303 7.5952×10^{-7} 1.6121×10^{-6}
	4[309], 4[556], 5.3722 5.5288 12[875] 5.6229	4[830], 4[1537], 11.3181 11.6695 12[2460] 11.8756			2[670], 2[1181], 2.9015 2.9263 6[1836] 2.9412	2[972], 2[1728], 3.9156 3.9363 6[2700] 3.9489
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$4_3[3^4]10_6[3^{10}]_{II}$	0.4958 ± 0.0796 1.3909×10^{-4} 8.0344×10^{-5}	0.3020 ± 0.0381 3.9193×10^{-5} 6.3920×10^{-6}			0.7490 ± 0.0461 1.0834×10^{-4} 1.4492×10^{-5}	0.6751 ± 0.0199 6.2752×10^{-7} 3.7299×10^{-7}
	4[325], 4[325], 5.3292 5.3292 12[532] 5.4737	4[866], 4[866], 11.5727 11.5727 12[1456] 11.8187			2[748], 2[748], 2.8369 2.8369 6[1187] 2.8711	2[1061], 2[1061], 3.8624 3.8624 6[1704] 3.8920
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$4_2[3^4]8_2[3^8]$	0.5102 ± 0.0568 4.1767×10^{-5} 3.0853×10^{-5}	0.3183 ± 0.0405 -9.0691×10^{-6} 7.8890×10^{-6}			0.7202 ± 0.0493 -1.1640×10^{-4} 2.1475×10^{-5}	0.6953 ± 0.0307 1.5917×10^{-5} 1.8233×10^{-6}
	4[259], 4[385], 5.2741 5.4026 12[535] 5.4916	4[683], 4[1040], 10.3309 10.5096 12[1469] 10.6331			2[613], 2[884], 2.8189 2.8507 6[1203] 2.8728	2[864], 2[1260], 3.8356 3.8651 6[1728] 3.8854
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$4_3[3^4]8_3[3^8]_I$	0.4577 ± 0.0761 -2.5782×10^{-5} 8.4655×10^{-5}	0.2910 ± 0.0421 -1.6419×10^{-5} 7.8109×10^{-6}			0.6887 ± 0.0279 8.0593×10^{-7} 7.8808×10^{-7}	0.6532 ± 0.0340 -2.5506×10^{-5} 3.2110×10^{-6}
	4[195], 4[356], 5.3231 5.4944 12[487] 5.5647	4[519], 4[978], 10.2736 10.5562 12[1355] 10.6657			2[454], 2[797], 2.8282 2.8708 6[1074] 2.8864	2[642], 2[1144], 3.8349 3.8759 6[1550] 3.8903
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$4_3[3^4]8_3[3^8]_{II}$	0.5030 ± 0.0603 2.6397×10^{-5} 4.0377×10^{-5}	0.3166 ± 0.0334 -1.3936×10^{-5} 2.7675×10^{-6}			0.7385 ± 0.0255 -2.6605×10^{-6} 9.6100×10^{-7}	0.6718 ± 0.0369 1.8952×10^{-6} 3.1533×10^{-6}
	4[219], 4[419], 5.3151 5.4988 12[683] 5.6047	4[582], 4[1152], 10.2818 10.5799 12[1914] 10.7461			2[517], 2[947], 2.7505 2.8131 6[1505] 2.8505	2[711], 2[1332], 3.8312 3.8769 6[2145] 3.9030
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$4_3[3^4]8_3[3^8]_{III}$	0.4931 ± 0.0560 1.5620×10^{-4} 3.1433×10^{-5}	0.2927 ± 0.0320 3.0191×10^{-5} 6.4721×10^{-6}			0.7294 ± 0.0348 4.5084×10^{-6} 2.5226×10^{-6}	0.6730 ± 0.0289 -2.7067×10^{-5} 3.4615×10^{-6}
	4[212], 4[410], 5.3491 5.5268 12[594] 5.6061	4[567], 4[1133], 10.3069 10.6019 12[1665] 10.7267			2[503], 2[927], 2.8151 2.8652 6[1317] 2.8853	2[708], 2[1328], 3.8107 3.8630 6[1900] 3.8832
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$4_4[3^4]7_2[3^7]_{II}$	0.4761 ± 0.0432 -7.3074×10^{-5} 9.5057×10^{-6}	0.3174 ± 0.0396 3.7498×10^{-5} 6.2114×10^{-6}			0.7144 ± 0.0228 7.2116×10^{-6} 6.1119×10^{-7}	0.6440 ± 0.0371 2.2191×10^{-5} 3.8797×10^{-6}
	8[345], 12[486] 5.5710 5.6379	8[961], 12[1370] 10.0208 10.1241			4[767], 12[1063] 2.7901 2.8222	4[1070], 12[1500] 3.8579 3.8800

	Cell	Bond	cell	bond	vertex	edge
$3_3[3^3]12_6[3^{12}]$	0.5077 ± 0.0896 4.5180×10^{-4} 3.2202×10^{-4}	0.2640 ± 0.0333 2.5513×10^{-6} 3.2552×10^{-6}			0.8273 ± 0.0209 1.3947×10^{-5} 1.6592×10^{-6}	0.7269 ± 0.0313 -2.2183×10^{-6} 1.6715×10^{-6}
	4[154], 4[223], 4.5065 4.7444 4[374], 12[520] 5.0214 5.1654	4[347], 4[529], 13.9135 14.2949 4[939], 12[1343] 14.7178 14.9248			2[359], 2[507], 2.8078 2.8402 2[827], 6[1133] 2.8730 2.8914	2[504], 2[720], 3.8333 3.8611 2[1188], 6[1638] 3.8889 3.9048
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$4_4[3^4]7_2[3^7]_{\mathrm{I}}$	0.4774 ± 0.0624 -1.6958×10^{-4} 4.8309×10^{-5}	0.3140 ± 0.0390 3.5112×10^{-5} 9.6223×10^{-6}			0.7305 ± 0.0263 7.4362×10^{-6} 1.0424×10^{-6}	0.6504 ± 0.0349 1.7310×10^{-5} 3.0145×10^{-6}
	8[403], 12[554] 5.4591 5.5379	8[1100], 12[1534] 10.0236 10.1213			4[1050], 6[1421] 2.4686 2.4828	4[1296], 6[1764] 3.8704 3.8889
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$5_4[3^5]7_4[3^7]_{\mathrm{I}}$	0.5112 ± 0.0412 -5.3542×10^{-5} 8.4833×10^{-6}	0.3406 ± 0.0207 2.1641×10^{-6} 5.0113×10^{-7}			0.7029 ± 0.0507 -5.7814×10^{-5} 1.4169×10^{-5}	0.6645 ± 0.0300 9.3355×10^{-6} 1.5205×10^{-6}
	4[173], 4[450], 5.3179 5.5733 12[729] 5.6653	4[460], 4[1254], 9.4391 9.7895 12[2065] 9.9090			2[411], 2[1003], 2.7932 2.8654 6[1590] 2.8931	2[574], 2[1437], 3.8188 3.8859 6[2300] 3.9096
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$3_1[4^3]5_1[4^5]_{\mathrm{I}}$	0.4003 ± 0.0534 -1.4626×10^{-5} 1.5617×10^{-5}	0.2368 ± 0.0198 -5.2284×10^{-6} 3.5502×10^{-7}	0.5786 ± 0.0400 3.9800×10^{-5} 5.2273×10^{-6}	0.4595 ± 0.0398 -2.9462×10^{-5} 5.1123×10^{-6}	0.6184 ± 0.0356 -1.2175×10^{-6} 2.6728×10^{-6}	0.5109 ± 0.0531 2.0112×10^{-5} 1.7919×10^{-5}
	4[508] , 6[795] 7.37kk01 7.4943	4[1872], 6[2979] 14.1731 14.3411	4[508], 6[795] 3.7835 3.8264	4[961], 6[1521] 6.1748 6.2406	4[569], 6[871] 3.5993 3.6739	4[1024], 6[1600] 5.6738 5.7387
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$3_1[4^3]5_1[4^5]_{\mathrm{II}}$	0.4065 ± 0.0646 -9.8055×10^{-5} 3.7012×10^{-5}	0.2309 ± 0.0241 2.2560×10^{-6} 6.6307×10^{-7}	0.5898 ± 0.0633 -1.8568×10^{-6} 3.3297×10^{-5}	0.4394 ± 0.0536 -1.3300×10^{-4} 3.6224×10^{-5}	0.6423 ± 0.0381 -5.1659×10^{-5} 7.7919×10^{-6}	0.5049 ± 0.0271 -9.8556×10^{-7} 8.8048×10^{-7}
	4[512], 6[648] 7.6328 7.6728	4[1954], 6[2486] 14.2027 14.2912	4[512], 6[648] 3.8750 3.8889	4[992], 6[1260] 6.2016 6.2349	4[545], 6[685] 3.7578 3.7839	4[1024], 6[1296] 5.8145 5.8349
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$5_3[3^5]8_6[3^8]_{\mathrm{II}}$	0.4631 ± 0.0471 2.8969×10^{-5} 1.3943×10^{-5}	0.3164 ± 0.0255 3.0266×10^{-6} 1.2108×10^{-6}			0.6800 ± 0.0384 -9.9185×10^{-6} 4.4523×10^{-6}	0.6587 ± 0.0246 1.5506×10^{-6} 6.3039×10^{-7}
	4[260], 4[675], 5.4308 5.6474 12[1095] 5.7205	4[706], 4[1906], 9.8669 10.1815 12[3132] 10.2854			2[608], 2[1490], 2.8191 2.8846 6[2368] 2.9071	2[857], 2[2149], 3.8273 3.8920 6[3442] 3.9140
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$5_3[3^5]8_6[3^8]_{\mathrm{III}}$	0.5052 ± 0.0474 -1.6178×10^{-6} 1.0710×10^{-5}	0.3211 ± 0.0274 -9.4980×10^{-6} 2.4172×10^{-6}			0.7145 ± 0.0299 8.1493×10^{-6} 1.7096×10^{-6}	0.6469 ± 0.0242 -1.0841×10^{-5} 7.2095×10^{-7}
	8[351], 12[814] 5.5328 5.6904	8[971], 12[2316] 10.0062 10.2366			4[804], 6[1782] 2.8458 2.8956	4[1144], 6[2580] 3.8531 3.9023
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$5_2[3^5]12_{12}[3^{12}]$	0.4897 ± 0.0718 -1.4823×10^{-4} 1.0656×10^{-4}	0.3037 ± 0.0257 -1.3145×10^{-5} 1.4342×10^{-6}			0.7384 ± 0.0258 3.3765×10^{-6} 8.4513×10^{-7}	0.6694 ± 0.0297 1.5898×10^{-5} 2.4608×10^{-6}
	4[192], 8[530], 5.1875 5.4981 12[803] 5.5890	4[498], 8[1457], 10.8072 11.2972 12[2244] 11.4332			2[442], 4[1156], 2.8371 2.8979 6[1724] 2.9165	2[627], 4[1675], 3.8405 3.9021 6[2514] 3.9204
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$5_4[3^5]7_4[3^7]_{\mathrm{II}}$	0.5001 ± 0.0342 1.7585×10^{-5} 3.4856×10^{-6}	0.3327 ± 0.0329 4.4678×10^{-5} 5.2573×10^{-6}			0.6973 ± 0.0235 -7.2307×10^{-6} 5.9288×10^{-7}	0.6688 ± 0.0303 2.4051×10^{-5} 2.0921×10^{-6}
	8[371], 12[541] 5.5364 5.6155	8[1027], 12[1519] 9.7254 9.8328			4[835], 6[1194] 2.8623 2.8844	4[1195], 6[1722] 3.8661 3.8885
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$5_3[3^5]8_6[3^8]_{\mathrm{I}}$	0.4867 ± 0.0459 -1.0891×10^{-4} 3.3218×10^{-5}	0.3265 ± 0.0237 -3.7362×10^{-6} 7.9808×10^{-7}			0.7108 ± 0.0200 2.8929×10^{-6} 4.2792×10^{-7}	0.6720 ± 0.0294 6.2044×10^{-6} 3.1424×10^{-6}
	6[214], 8[535], 5.3925 5.6150 12[790] 5.6835	4[577], 8[1502], 9.8024 10.1278 12[2245] 10.2254			2[506], 4[1192], 2.8261 2.8859 6[1728] 2.9051	2[731], 4[1762], 3.9398 4.0204 6[2573] 4.0459
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$4_2[3^4]12_6[3^{12}]$	0.5117 ± 0.0960 1.0580×10^{-4} 2.1578×10^{-4}	0.2637 ± 0.0295 -4.6675×10^{-6} 1.9508×10^{-6}			0.7861 ± 0.0215 7.1685×10^{-6} 4.3496×10^{-7}	0.7273 ± 0.0227 1.1750×10^{-6} 7.1142×10^{-7}
	4[200], 8[444], 4.6100 5.0495 12[629] 5.1924	4[461], 8[1121], 11.1714 12.1802 12[1633] 12.4850			2[474], 4[997], 2.8017 2.8646 6[1388] 2.8847	2[664], 4[1428], 3.8614 3.9048 6[2002] 3.9191

	Cell	Bond	cell	bond	vertex	edge
$4_2[3^4]18_{12}[3^{18}]$	0.5249±0.1168 1.1603×10^{-3} 6.1479×10^{-4}	0.2493±0.0632 7.1820×10^{-4} 2.0594×10^{-4}			0.8497±0.0234 5.2173×10^{-6} 6.1183×10^{-7}	0.7743±0.0483 8.6384×10^{-5} 1.2578×10^{-5}
	8[237], 4[428], 4.8608 5.1449 12[675] 5.3156	8[576], 4[1101], 15.0868 15.8656 12[1794] 16.3155			4[529], 2[929], 2.8696 2.9020 6[1441] 2.9216	4[774], 2[1376], 3.9664 4.0087 6[2150] 4.0344
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$5_3[3^5]7_3[3^7]_{\rm I}$	0.4882±0.0494 -4.4634×10^{-5} 1.5462×10^{-5}	0.3335±0.0310 -1.2425×10^{-5} 2.4243×10^{-6}			0.7149±0.0273 1.6306×10^{-6} 9.8464×10^{-7}	0.6495±0.0278 5.9525×10^{-6} 1.4733×10^{-6}
	4[170], 4[399] 5.3529 5.5739	4[455], 4[1112] 9.4110 9.7392			2[413], 2[907] 2.7797 2.8512	2[574], 2[1293] 3.8084 3.8716
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$5_3[3^5]7_3[3^7]_{\rm II}$	0.4345±0.0489 8.3482×10^{-5} 4.0009×10^{-5}	0.2930±0.0243 -9.3058×10^{-6} 1.1454×10^{-6}	0.4785±0.0354 4.7566×10^{-5} 7.5326×10^{-6}	0.3239±0.0187 -4.1986×10^{-6} 3.8717×10^{-7}	0.6900±0.0283 -7.8267×10^{-6} 1.5860×10^{-6}	0.6421±0.0289 3.4481×10^{-6} 1.8973×10^{-6}
	2[167], 2[61], 5.8443 5.2787 6[535] 6.2430	2[161] 6[1670] 9.8137 11.3257	2[167], 2[61], 5.7485 5.2131 6[535] 6.1234	2[480], 2[159], 10.4458 9.5849 6[1638] 11.0000	2[386], 2[161], 2.7720 2.6460 6[1124] 2.8665	2[535], 2[213], 3.8019 3.6901 6[1611] 3.8845
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$3_2[4^3]5_2[4^5]_{\rm I}$	0.4099±0.0264 4.7650×10^{-6} 1.0783×10^{-6}	0.2271±0.0154 2.1524×10^{-6} 1.2777×10^{-7}	0.5687±0.0469 2.6238×10^{-5} 8.5056×10^{-6}	0.5028±0.0283 1.1978×10^{-5} 1.1260×10^{-6}	0.6228±0.0346 -1.1730×10^{-5} 5.5785×10^{-6}	0.5022±0.0343 3.4916×10^{-5} 3.2050×10^{-6}
	2[277], 2[379], 7.3285 7.4248 6[497] 7.4970	2[1407], 6[1863] 14.0640 14.1857	2[277], 2[379], 3.7473 3.7836 6[497] 3.8109	2[519], 2[717], 6.0308 6.1004 6[947] 6.1521	2[325], 2[435], 3.5446 3.6046 6[561] 3.6506	2[576], 2[784], 5.6285 5.6811 6[1024] 5.7207
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$3_2[4^3]5_2[4^5]_{\rm II}$	0.4322±0.0512 5.8373×10^{-5} 1.9444×10^{-5}	0.2344±0.0171 -1.5204×10^{-6} 2.2157×10^{-7}	0.5638±0.0590 4.8226×10^{-6} 1.5253×10^{-5}	0.4749±0.0440 -5.1510×10^{-5} 1.0399×10^{-5}	0.6260±0.0368 -2.9282×10^{-5} 6.7165×10^{-6}	0.5468±0.0270 8.1058×10^{-6} 1.6103×10^{-6}
	2[276], 2[496], 7.1812 7.3831 6[780] 7.5051	2[1831], 6[2927] 14.1038 14.2890	2[276], 2[496], 3.6739 3.7540 6[780] 3.8026	2[507], 2[931], 5.9961 6.1267 6[1483] 6.2036	2[313], 2[545], 3.6422 3.7284 6[841] 3.7812	2[570], 2[1016], 5.6702 5.7520 6[1590] 5.8013
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$3_3[4^3]5_3[4^5]$	0.5029±0.0759 8.6335×10^{-6} 1.4737×10^{-4}	0.2504±0.0233 -3.9666×10^{-6} 8.6134×10^{-7}	0.6238±0.0601 -1.3226×10^{-4} 3.8526×10^{-5}	0.5029±0.0359 1.1319×10^{-4} 1.9717×10^{-5}	0.6645±0.0414 1.1038×10^{-4} 1.6270×10^{-5}	0.5801±0.0306 -3.3375×10^{-5} 3.6510×10^{-6}
	4[161], 2[315], 6.9565 7.2317 6[473] 7.3700	2[1139], 6[1743] 13.8191 14.0425	6[161], 2[315], 3.5404 3.6698 6[473] 3.7294	4[285], 6[578], 5.8175 6.0104 6[882] 6.1020	4[200], 2[369], 3.3600 3.5122 6[539] 3.5918	4[336], 2[648], 5.4881 5.6204 6[968] 5.6880
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$3_1[4^3]6_2[4^6]_{\rm I}$	0.4339±0.0259 4.6901×10^{-7} 6.9917×10^{-7}	0.2335±0.0200 -3.2315×10^{-6} 3.6617×10^{-7}	0.5468±0.0321 3.4990×10^{-5} 4.5289×10^{-6}	0.4235±0.0152 2.5190×10^{-6} 1.5119×10^{-7}	0.6418±0.0638 -1.1180×10^{-4} 4.6849×10^{-5}	0.5715±0.0431 4.9165×10^{-5} 7.7330×10^{-6}
	2[760], 6[1190] 7.5263 7.6202	2[2860], 6[4534] 15.2503 15.4027	2[760], 6[1190] 3.8342 3.8672	2[1457], 6[2301] 6.7138 6.7718	2[833], 6[1281] 3.6879 3.7471	2[1536], 6[2400] 5.7513 5.8008
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$3_1[4^3]6_2[4^6]_{\rm II}$	0.4740±0.0681 -1.5376×10^{-5} 4.0570×10^{-5}	0.2803±0.0276 -1.1319×10^{-5} 1.3703×10^{-6}	0.6333±0.0534 -1.4476×10^{-4} 2.7149×10^{-5}	0.4785±0.0260 1.8163×10^{-5} 1.3662×10^{-6}	0.6961±0.0552 2.5890×10^{-5} 2.2393×10^{-5}	0.6132±0.0465 6.0868×10^{-5} 1.1922×10^{-5}
	2[245], 2[477], 7.0531 7.3040 6[715] 7.4294	2[1742], 6[2656] 14.8576 15.0715	2[245], 2[477], 3.6408 3.7400 6[715] 3.7874	2[446], 2[892], 6.3184 6.5135 6[1354] 6.6041	2[284], 2[532], 3.5493 3.6541 6[782] 3.7136	2[504], 2[972], 5.6349 5.7243 6[1452] 5.7741
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$3_2[4^3]6_4[4^6]$	0.4540±0.0687 7.8079×10^{-5} 4.6624×10^{-5}	0.2265±0.0369 -1.5535×10^{-5} 2.8752×10^{-6}	0.6160±0.0685 -1.8755×10^{-4} 5.5163×10^{-5}	0.4908±0.0346 -1.2315×10^{-5} 2.5752×10^{-6}	0.6320±0.0532 1.0557×10^{-4} 2.0627×10^{-5}	0.5527±0.0434 1.0264×10^{-5} 1.0088×10^{-5}
	2[203], 2[440], 6.9754 7.2955 6[550] 7.3673	2[1605], 6[2026] 14.9533 15.0642	2[203], 2[440], 3.6059 3.7318 6[550] 3.7600	2[366], 2[821], 6.4645 6.6358 6[1034] 6.6731	2[238], 2[491], 3.5294 3.6660 6[607] 3.6969	2[420], 2[900], 5.8095 5.9689 6[1122] 6.0036
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$3_3[4^3]6_6[4^6]$	0.4042±0.0580 3.6150×10^{-5} 1.8890×10^{-5}	0.2281±0.0277 2.6810×10^{-6} 7.5233×10^{-7}	0.5958±0.0567 -4.2796×10^{-5} 1.4656×10^{-5}	0.4698±0.0313 3.1642×10^{-5} 4.6775×10^{-6}	0.6353±0.0449 -2.0589×10^{-5} 8.7591×10^{-6}	0.5210±0.0370 7.9238×10^{-6} 2.4104×10^{-6}
	2[230], 2[490], 6.9391 7.2735 6[656] 7.3720	2[1782], 6[2418] 14.8911 15.0488	2[230], 2[490], 3.5304 3.6816 6[656] 3.7256	2[406], 2[902], 6.5714 6.7095 6[1222] 6.7496	2[267], 2[543], 3.5955 3.7127 6[717] 3.7490	2[480], 2[1008], 5.6417 5.7500 6[1344] 5.7827

	Cell	Bond	cell	bond	vertex	edge
$3_1[4^3]8_4[4^8]$	0.4525 ± 0.0780 7.1913×10^{-4} 1.4678×10^{-4}	0.2122 ± 0.0230 -9.6553×10^{-6} 8.2295×10^{-7}	0.5237 ± 0.0627 -1.0143×10^{-5} 3.2798×10^{-5}	0.3807 ± 0.0401 3.5308×10^{-6} 5.4378×10^{-6}	0.7010 ± 0.0470 -7.5034×10^{-5} 1.2495×10^{-5}	0.6086 ± 0.0679 1.6014×10^{-4} 3.8208×10^{-5}
	2[238], 2[490], 7.0336 7.3224 6[708] 7.4350	2[1794], 6[2632] 16.9197 17.1071	2[238], 2[490], 3.6555 3.7592 6[708] 3.7994	2[435], 2[921], 7.3931 7.5831 6[1345] 7.6550	2[288], 2[561], 3.4028 3.5651 6[793] 3.6318	2[490], 2[1000], 5.4898 5.6420 6[1440] 5.7014
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$3_1[5^3]4_2[5^4]_{\mathrm{I}}$	0.4047 ± 0.0637 6.1235×10^{-5} 3.1306×10^{-5}	0.2185 ± 0.0212 5.5289×10^{-6} 5.4838×10^{-7}	0.6180 ± 0.0491 6.5660×10^{-5} 1.1080×10^{-5}	0.4930 ± 0.0327 1.4895×10^{-5} 2.1733×10^{-6}	0.5516 ± 0.0469 4.3305×10^{-5} 1.3841×10^{-5}	0.4101 ± 0.0301 1.2816×10^{-5} 2.1196×10^{-6}
	2[252], 2[570], 8 8.3298 6[777] 8.4247	2[2374], 6[3273] 16.0126 16.1577	2[252], 2[570], 3.7143 3.8105 6[777] 3.8378	2[468], 2[1086], 6.3932 6.5157 6[1491] 6.5500	2[281], 2[613], 4.4982 4.6591 6[827] 4.7062	2[632], 2[1428], 7.4525 7.6345 6[1946] 7.6865
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$3_1[5^3]4_2[5^4]$	0.3457 ± 0.0484 8.5340×10^{-5} 2.5074×10^{-5}	0.1820 ± 0.0200 -5.3816×10^{-6} 6.0154×10^{-7}	0.6226 ± 0.0443 -5.2469×10^{-5} 1.2122×10^{-5}	0.5695 ± 0.0477 -1.2139×10^{-4} 1.8902×10^{-5}	0.5527 ± 0.0393 -2.9005×10^{-5} 1.0732×10^{-5}	0.4162 ± 0.0419 -3.2254×10^{-5} 7.5538×10^{-6}
	8[213], 2[380], 8.9202 9.1842 6[595] 9.3445	6[950], 2[1745], 16.8147 17.2218 6[2780] 17.4626	8[213], 2[380], 3.1080 3.1632 6[595] 3.1966	8[331], 2[601], 4.4411 4.5324 6[951] 4.5868	8[169], 2[289], 4.2604 4.4291 6[441] 4.5351	8[360], 2[640], 7.3444 7.5062 6[1000] 7.6040
	Cell	**Bond**	**cell**	**bond**	**vertex**	**edge**
$3_2[5^3]4_4[5^4]$	0.3672 ± 0.0405 -6.6453×10^{-5} 1.1174×10^{-5}	0.1766 ± 0.0243 -3.2608×10^{-6} 1.0035×10^{-6}	0.6265 ± 0.0398 7.1192×10^{-5} 7.0957×10^{-6}	0.5822 ± 0.0505 2.3922×10^{-5} 2.8516×10^{-5}	0.5413 ± 0.0515 -9.8715×10^{-5} 1.9319×10^{-5}	0.4101 ± 0.0536 9.3126×10^{-5} 3.1517×10^{-5}
	4[168], 4[270], 8.7143 8.9778 6[396] 9.1515	2[732], 2[1212], 16.5929 16.9868 6[1812] 17.2395	4[168], 2[270], 3.0357 3.0963 6[396] 3.1364	4[255], 2[418], 4.4078 4.4928 6[621] 4.5475	4[141], 4[216], 4.3688 4.4907 6[307] 4.5733	4[308], 4[485], 7.2662 7.4144 6[702] 7.5128

Table 3.1 *Percolation statistics of the 2-homohedral tilings. Each item of the statistics are, from top to bottom, mean ± standard deviation, second moment, and third moment. Similarly each of the items in each box contains the number of runs [the number of network components] and the coordination number.*

The advantage of sparse- over dense networks described by Burt (1992) is an idea similar to that of the benefit of decentralisation and local autonomy in politics. According to him, size still matters but the cost of maintaining the size is also important. The idea can be very general; dense networks are virtually worthless monitoring devices, while sparse networks give more information benefits. Taking opportunity costs into account, the rate of return of a dense network is lower than that of a sparse network. The idea can be applied to a strategic network expansion which is crucial for sales persons and jobs hunting alike.

Something similar to this comes up in a variety of fields. In Sociology it is sometimes called the *strength of weak ties*. A tie between two things is weaker the less each of them has in common with the other. It is well known, *i.e.* the origin unknown, that best friends are usually those who have least in common with each other. It would be interesting to trace this idea back to where it first appeared in literature.

§ 3.5 Economic modelling

My interest in the modelling of economic systems started immediately following the devaluation of the thai Baht in July 1997, which in turn triggered the economic collapse of east Asian countries. But the seed of this interest may have been germinated nearly seven years earlier when I went to Budapest for a job training through a program by the *Association Internationale des Etudiants en Sciences Economiques et Commerciales*, aka AIESEC. It was there that I experienced firsthand an economic system with inherent potential, but which was ailing and struggling to survive. It was also there that I have come to know the word *hard currency* and learnt about terrible consequences to people in a country whose money is otherwise from being hard. Some of these consequences are the proliferation of black markets where money is illegally traded for hard currencies like the US Dollar or Deutsch Mark, and the ever soaring inflation rate. In Poland at that time, for example, the inflation is above 700 percents. In such an unstable society stealing is a matter of fact in everyday life. Robbery becomes so widespread that the police no longer bother to catch robbers, and this is so much a more decent thing to do since otherwise there would have been no place to put them in in the first place. This I know because it was in Poland around June 1990 that I was robbed of nearly all of my personal belongings. Afterwards when I have become initiated to the theory of percolation, I ascribe the transition between a hard and a soft currency to the percolative process. Now I still see this, as well as the changeover of the eastern bloc countries, as a phase transition.

My brief presentation at a conference in Japan (Tiyapan, 1997) has been warmly received by some of the experts who attended, despite criticism from a member of the organising committee before the presentation. My belief was, and still is, quite different from the conventional theory as to the cause of this national mishap. Instead of seeing the attack on thai Baht made by an american hedge fund which is run by George Soros, I put most of blame on other seemingly unrelated factors that deprive Thailand of essential robustness in many of its fundamental structures, particularly on too much centralisation towards Bangkok and the percolated traffic networks in that city which came close to a standstill. Others argue that the Soros hedge fund represents a corporate crime against a nation because no matter whether it is the true cause of the catastrophy or not the intention of attack was there. But such is not the kind of matter which concerns us here. In essence, I give all the value to the structural soundness of a system and the infrastructures and none to any single occurence or mistake in management decision. A country, I argued, will run sound if if it has a potential to do so, and not otherwise. The potential of a country depends on such things in the background as all the infrastructures including transportation and telecommunication systems, education, forest, and decentralisation.

The self-sufficiency practice recommended by King Bhumibolof Thailand in his annual speech to the thai people on his birthday in the December following the crisis is the very core idea of decentralisation in disguise, namely that of local management which will ultimately lead towards local government. In his speech he told the Thai people to live a self-sufficient life, to grow vegetables in the backyard enough for one's own need, and to sell and make a business out of it if one has more. I believe his idea goes deeper and his teaching more a parody from the necessity from him to remain impartial even in the face of a crisis. In short, the self-sufficient of oneself will lead to that of one's family, village, town, district, province, and finally the country. The Constitutional Laws have been changed to favour this, which I think is a crucial step.

There has been a complete turnaround in the policy in Thailand now compared with what had been before the 1997 crisis. Then it was everything to be centralised in Bangkok. Every prosperity, you name it, must go to the capital, and every policy out from it. It had been so for a long time, for many generations, and democracy did not much improved it. When Smyth (1898) lived in Thailand in the 1890's he saw what I had seen one century later in the 1990's, that everything beautiful in Bangkok comes from, and at the expense of, the rest of the country which is left to be poor. Anyone travelling in the city even today can see quite easily that Bangkok by no mean represents Thailand. With this kind of policy, no doubt everyone tries to get to Bangkok. This worsen its problems further. Nothing that I may say on this point can be better than the following account made by Smyth from the work already quoted.

> I am certain that the Government in Bangkok have never realised the disastrous effect which the system had on the distant province. The revenue each year was maintained at the proper figure, and therefore no questions were asked. The money went to Bangkok, and the P. W. D. spent it in improving the look of the town. Visitors were impressed with what was being done in Bangkok, and went away persuaded that Siam was an enlightened country. The poverty-sticken condition of the provinces did not come under their notice. 'That's all our money,' cried a disconsolate provincial Government to me, spreading wide his fingers towards a canal in Bangkok beeautifully embanked, with a charming road on either side. 'I have had to roof my court-house at my own expense.'

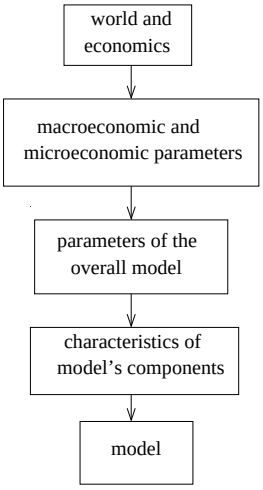
Fifty years on, the canals in question started to deteriorate and fifty years later either have been filled to make more roads or left a dirty sewer. In 1992 there was a plan made by the Bangkok Metropolitan Administration to construct wastewater treatment plants around Bangkok. The construction was to be in two phases each of which worths 5,000 bahts, and equivalent of 120 million pounds sterling. At that time I was working at the Loxley International Public Company in Bangkok in the capacity of a sales engineer. In the second bidding for this turn-key project I was transfered to the Business Development Division to work as the estimator of the project, reporting directly to Mr. Ronald James Savage who was one among the Board of Directors of the company. Loxley joined force with several consultants and contractors to bid for this project, and as the company was to provide all the necessary equipments and technologies my job was to search all corners of the world for everything everyone would need. It was chalenging, but then I was a workaholics so it was all I needed. In the end we did not win the project, however, so that is as much as I have to say about it except for one thing, that is that I think the wastewater treatment project came far too late and was still not completed when the 1997 crisis occured. Seemingly unrelated, in the present argument and together with other things such as traffic, education, decentralisation, it has got everything to do with that crisis.

Now the policy has turned into that which lets the remote parts of the country manage their own funds and make their own decisions. There is, for example, a scheme of *a product a village* that fosters local artisans. As a result, local heritage becomes fad where is only five years ago it was fiction, something to forget, not to think about. If there is no hiccough in this process or a *coup d'état* of some sort, then the policy of Thailand now is by far sounder than it ever had been. Therefore Thailand to me now is in a better shape than it used to be when it was still dubbed a *tiger or the East*. However people often think otherwise. I always find myself involved in a controversial or difficult issue, and on the wrong side thereof. Before the crisis, for instance, when most people looked up to America with hopeful eyes I always said to my colleagues, 'we should not put our whole faith in America', that Thailand should not rely too much on any other countries, and I was looked at questioningly. After 1997 I said, 'it's not half the fault of the Americans or anyone else but us', and I was eyed with disgust. The hedgefund which attacked the Thai Baht in 1997 was only an instrument, not the reason, for what happened thereafter. It answer the *when* question, but the more important *why* question has already been answered long before that (*cf* Smyth, 1898). I can say this because as easly as 1990 while the Thai economy was in the bull I wondered myself why it has not collapsed because, as I often said when discussing with my colleagues engineering students, according to my reasoning it should. Immediately after the disastrous event while the country was in the blue and I was in Japan I told myself that I had already known why it happened, now I knew *when* and *how* it did.

What followed was a period of a few years when the whole country, noticeably Bangkok, was so quiet you would not believe. For some times the country was a paradise for travellers. Everything was cheap and nothing was not on sale. Instead of increasing, I felt that on the whole crimes had reduced, perhaps because such tragic events always shook the hearts of people, the thieves' and the gentles' alike. The Tourism Authority of Thailand coined this marvellous word *Amazing Thailand*, to promote tourism in Thailand abroad, which fitted the mood of the country like a glove because everyone felt like staring wide-eyed with disbelief at what had happened and what was happening around them. I was equally amazed at what was happening around me when I visitted Thailand from Japan, but not at what had happened because, like the next great Tokyo earthquake, I had been expecting it for quite a while before I got tired and stopped thinking about it.

An analogy of distributed government can be seen in computer science under such terms as distributed systems, parallel processing, and object-oriented programming. I see it as essential, if not as the only way, towards a sound and robust economic systems.

At this ATACS conference in Tokyo in October 1997 I explained economic modelling as follows.



The economics of the world can be represented by macroeconomic and microeconomic parameters. Together these two make up the parameters of the overall model and describe the characteristics of components of the model.

Figure ℵ Economical models.

Later that year I submitted two papers (Tiyapan, 1997) electronically to a journal, the American Journal of Physics I think it was. For the reason that both papers were products of my personal research interest which has nothing to do with my commitment to the Tokyo Institute of Technology, I could only spend a meagre amount of time on them. As a result, the work was too preliminary and there was no reply from the publisher except the first computer-generated message saying that they have received my files. No matter how poor my papers were, I felt that to get a negative feedback from an editor was already bad enough, but to get no reply was worse.

In the light of the argument above, what Hammond (1998) called a *Market World* corresponds to the case where the globalisation expands smoothly and permeates all countries from developped to third world to form a single unbroken cluster. What he called a *Fortress World* corresponds to the world where the the free market only benefits some countries while the rest is left deprived. The poorer countries encroach on the richer ones like water encroaching the residual oil in an oil reservoir; islands of prosperity surrounded by seas of poverty. Yet his third and last vision or version of the world, the *Transformed World*, would correspond to a changed structured of the network of culture and economy which gives a more robust system and accommodates all the needs of the future generations. Among the things that are changed in this case are the cultural norms and values.

Cultures are like languages. Like switching from one language to another, when travelling to another country one switches from one frame of mind into another. Unless this is properly done there will always be some adverse symptoms to experience, and *culture shocks*, which are common in novices, become less so in experts. In the languages of cultures, when we deal with the Japanese's we should keep as much a polite manner as possible and take our responsiblilities to the other party as seriously as possible regardless to what the small prints might or might not say. On the other hand if we are to conduct our business with the Americans, for instance, we should question every move. Generally speaking while the West favours contracts, the East has been run mainly by personal relations. The catch came when the latter party adopted so many of the ways and ideas of the former, and these got improperly mixed together with its own ways of thinking which are different to produce an incoherent, unstable and insecure structure. Stevenson (1895) when travelling *out of himself* to the United States in 1879 came across a mexican who was scruple enough to knowingly sign an imperfect paper for fear that the other party might think that he doubt his words otherwise. The scruple sounds odd, he said, because we "have been brought up to understand all business as a competition in fraud, and honesty itself to be a virtue which regards the carrying out, but not the creation, of agreements." This no doubt answers the question why the 1994 economic crisis in Mexico occurred.

Because there is more to the economic well being of a country than economic figures can tell, one needs to look around in order to to find clues to what is going wrong. For this, the richest source is in the literary literatures, some of which had seeped out through the unconscious minds of some of the most experienced and sensitive minds. Also, at least in the field of economics the study of science and the social study should go along side-by-side. More than economic figures and those things which meet the eyes, studies in this area should take into account hidden factors, for example education, centralisation and policy in general, and turn them into a form which can be numerically or theoretically studied. Centralisation always adversely effects the economics, for

instance the greater control by the central state of government expenditure as well as growth of the public sector Germany immediately after the First World War is followed by the hyperinflation which began in summer 1922 (*cf* James, 1986).

Descriptions of economic crises usually resemble percolation, eventhough or especially when they come from someone who has no ideas of percolation in his mind. For example, Rodelet and Sachs (2000) think that small events at times have large consequences. They talk about crisis of success, that both the collapse of Mexico in 1995 and of Korea in 1997 came after a prolonged period of euphoria. In Thailand, for instance, indicators in the third quarter of 1997 suggested nothing like a financial meltdown. On the other hand, while Mlesi-Ferretti and Razin (2000) mention a 'perfectly forseen speculative attack' as the thing which triggered the crisis in this case, there is no question about the network having already greatly weakened before the attack.

One has to distinguish things which trigger from things which cause. One can be very much certain that crises of this scale must come from a very slow process which works on the structure within. It may be true that delaying bubbles from bursting this way even results in a deeper crisis afterwards.

§ 3.6 Cosmology

The universe is made up of superclusters of galaxies, each of which contains local clusters. Each local cluster contains galaxies, each of which contains stars. When stars explode, gas bubbles are formed which expand to meet one another, forming walls of materials in the manner of the Voronoi tessellation in three dimensions. Within the planets around each stars there exist yet other structures which can be similarly represented by the Voronoi tessellation.

Superclusters are like endless cobwebs covering the space. The Andromeda galaxy, the Sculptor group, the Virgo cluster, and the M81 group are examples of the members of the Local Supercluster in which our Milky Way galaxy reside. The Pisces-Perseus supercluster is 250 million, and the Hercules supercluster 500 million light-years away from us.

Assuming the big bang origin of the universe, the defects originated in the phase transition at the Big Bang could play an important role in the formation of galaxies and their clusters and superclusters. Tom Kibble predicted this in 1976 (*cf* Croswell, 1995), and gave three types of possible defects, namely the point-like monopoles, the line-like cosmic strings, and the plane-like domain walls.

Van de Weygaert (1989) studied the statistics of 3-d Voronoi tessellations for the purpose of understanding the structure of the universe. Here *neighbours* are defined by a common polygonal face whereas *full neighbours* have an additional requirement that the line joining nuclei intersects this common face. Taking a linear section through d-d, $d \geq 2$, the mean length is

$$\langle \lambda \rangle = \frac{d\Gamma\left(d-\frac{1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)^2}{(d-1)!\rho^{\frac{1}{d}}2\Gamma\left(\frac{d}{2}+1\right)^{2-\left(\frac{1}{d}\right)}\Gamma\left(2-\frac{1}{d}\right)}, \qquad (\mathbf{1})_{\text{iii}}$$

where ρ is the number density, in other words nucleus density.

In three dimensions we need at least two points in order to establish the distance of an object by a geometrical method. Because the cosmic scale is so large compared with the scale of the solar system and Earth, all the observation points we may choose become the same and only one point in practice. Therefore the determination of the distance to extragalactic objects, which is crucial in cosmology, can only be achieved by the various means of observations from a single point in space. This is a limitation that has caused the establishment of distance to be a hot, long-standing controversial issue. As the universe is expanding with the receding velocity of objects increases with their distance away from us, and as this velocity can be accurately determined by the amount of redshift in the light of these objects observed, the study of distance becomes the study of a single, universal constant called the Hubble constant. This is only a constant in theory not in practice. All numerical values of the Hubble constant used in literature are fictitious to some degree. If its exact value is known, then the accurate distance to any cosmic object is simply its radial velocity, *i.e.* its velocity away from us, divided by this Hubble constant. The reciprocal of the Hubble constant is called the Hubble time. The importance of the Hubble constant, together with the difficulty in finding it, make the observational and the theoretical parts of cosmology inseparable from each other. They are also the cause of the proliferation of the modern literature on Astrophysics. One example of studies of the extragalactic distance scale is that reported by de Vancouleurs (1993) which includes among the objects studied the following objects belonging to the Local Group, namely the Large

Magellanic Cloud, M31, M33, NGC 3109, as well as the objects beyond the Local Group, namely NGC300, IC4182, NGC2403, NGC5128, M81, M101, M104 (NGC4594), and NGC4571.

§ 3.7 Philosophy

Philosophy precedes science, and so the case is with Voronoi tessellation by more than two hundred years. In his philosophical scientific work, Descartes (1644) imagines the universe as a tessellation of vortices surrounding stars. According to the pictures he gave, a few of which are duplicated many times over, the positions of these stars are random. Among these, the most abundant picture resembles a two dimensional picture of biological cells or a Voronoi graph. His idea is that stars always position themselves as far apart from one another as possible. At any instance each of these bodies resides within a partition of its own, where it acts as the centre of a vortex around which everything inside the partition revolves.

However, when one would like to track the movement made by a body one may consider the rest as having a fixed position. Starting from anywhere within a partition, the body under consideration moves towards a boundary that lies across its trajectory. Once the boundary is reached, it continues moving along that boundary until it reaches a vertex from the requirement of keeping itself furthest from surrounding stars, and then it chooses the one of the two choices of boundary branches which would make it deviate the least from its present direction. In this way a body snakes its way further and further from its starting position. Although the pictures that Descartes gave were in two dimensions, there is no doubt that he means, and was talking about a three dimensional space. Therefore these two dimensional pictures of his can either be a 2–d section of a three dimensional space, or simply a 2–d representation of it.

As is the case with the study of incompressible fluid flow by visualising it as a displacement of small rigid spherical particles, whose treatment within the same treatise of his predates the ground-breaking theoretical works of Hele-Shaw (1899) by more than two and a half centuries, the idea he has given about the movement of heavenly bodies predates the idea of van de Weygaert where the similarity is obvious in both the treatment of, as well as the objects considered. Van de Weygaert and Icke (1989) considers the movements of galaxies within voids with sizes several orders larger than themselves. In their simulation, each galaxy moves outwards from anywhere within a void until it reaches a wall where, because of the interaction with other galaxies coming from a neighbouring void which it meets, it loses its radial velocity and continues to move with reduced speed along the plane of the wall. Eventually it will come to an edge where it loses a second part of its velocity components, again thanks to other galaxies this time coming from the other two of the three faces which comes together at the edge. The last component of velocity it has left then takes it as far as the first vertex it should meet and no further. Having lost all the remaining energy to travel further, again through the encounter with other galaxies this time from three other edges which meet with the edge along which it has been travelling, it joins other galaxies to form a cluster which keeps growing all the time.

If the idea of Descartes is that of stars travelling along a Voronoi graph, then the idea of van de Weygaert and Icke represents a universe which is lumpy at Voronoi vertices. Seemingly different, both ideas are very similar. Both consider partitions containing nuclei. The only difference seems to be that in the former case these nuclei are massive bodies, while in the latter they are voids. Thus in a way the two can be said to be dual to each other. Furthermore, the scale being considered seems to be by many orders different. Descartes' vortex idea can easily apply to the scale size of a galaxy, while the vertex idea proposed by van de Weygaert and Icke seems to agree well with observations made in scale unimaginably large compared with that of the other, since they obviously consider giant clusters of over a thousand galaxies in size as being vertices, in other word points.

Voronoi tessellation thus came up philosophically first and then science. Percolation theory, on the other hand, is itself philosophical. Not only that it often sheds light on the underlying reason of many things, but it sometimes is the core philosophy in its own right.

Karma is the central philosophy of Buddhism. In East Asia there are similar sayings which mean *Karma is deed* (*cf Lan. Kammṅ pẹn kạrkdaṃ* and *Th. Kamr gụe kạrkradaṃ*). The word *karma* in English adopts a more practical one among the plethora of implications or nuances in meaning, which says that it is the sum of actions of a person in one of his successive lives, which decides his fate in the next. Another meaning puts it simply as destiny or fate. Karma and enlightenment are the two things around which Buddhism philosophy revolves. Karma entails enlightenment. In other words, the latter is a subset of the former. To put it plainly, enlightenment is Pcwhile karma is the corresponding Pc-process.

To me the teaching of Buddha is about percolation, and hence the idea was discovered by Buddha whom I consider a great philosopher, at least more than two millenia and five centuries before it was rediscovered in Physics of our time. Now it may seem ironic after having said that the Buddhist philosophy is percolation if I as professedly a christian should say that I have converted myself to the religion as a result of a percolation, having been among buddhists since I was born and having led my life as one for twenty years. When I left a job with a good prospect behind me to come to England in 1994 I said to myself that having been forsaken by two of the women I most love within a period of four years it did not seem that life was no longer worth living and making money seemed all very well except that I had no one now to make it for. Then in the sombre atmosphere of Manchester I read the two paperback translated volumes of *Les Misérables* by Victor Hugo which suddenly turned me into a christian. I think of this experience now as the percolative workings of various elements. Having never been the best student in my class, except sometimes in the subject of the English language, I had found myself often in ungraceful academic situations from 1984 to 1991 when I received my first degree. My study had given me much stress and the life in Bangkok with its best percolated traffic jam in the world was no help. Due to some historical reasons and the difference in the languages a lanna in Bangkok is usually considered as belonging to a second class population. This combined with the heart having been broken from the cause mentioned and from the background of having lived with a Kiwi western family and the fact that I have always studied in a christian school which had provided me with the necessary background and belief, it was not difficult for me to profess myself a christian, but Hugo still has the final credit for it. I spent close to twelve years studying at the Montfort College in Chiangmai, Thailand. The college was named after Saint Louis Marie Grignion de Montfort (1673–1716) who founded the Congregation of the Brothers of Christian Institute of St. Gabriel in 1705 to which my school belongs.

I still believe in the philosophy of Buddha. I see Buddhism as a philosophy the idea of which is an analogy to percolation, but I do not see it as a religion, which to me necessarily must start from a myth and have a god. Buddha, however, is no god, and taught his findings which were results of the Enlightenment he experienced as a philosophy from the start. What myths there are in Buddhism now have invariably been added hundreds if not thousands of years later in accord with the tastes and methods of the rulers and preachers. I do not consider this kind of myths as real, and consider them superficial and unnecessarily for as good a philosophy as Buddhism.

In doing a PhD I find that I have to make war with my own ignorance and unproductiveness. But no doubt at each step forward and as each new idea comes in I am a step closer to my goal. And if I think that I should mind more about the quality of my work than about the end result, then I am using the same argument that I use in § 3.7 to explain the economic potential of a country. The internet and plastic money or credit cards may be factors which help the birth of the Euro possible. But the potential and connectivity has already gathered up considerably since the fall of the Iron Curtain and the reunification of Germanywith its amazing one to one exchange from the eastern to the western Deutsch Marks. As much as that the fall of both the Iron Curtain and the Berlin Wall may owe much to the internet and the spread of information by the media the internal flaws within the communist regimes of the Eastern Block countries must have also played a great part. Because otherwise why the same has not happened to the West saturated with the media? The West with its distributed government, a system inherited from America, has an advantage of more robust a system. This is a strong point of democracy. Socialism also has its unique strong point regarding the feeling of security in individuals which helps making the system robust. So as the flaws inside a percolating cluster collectively break it down, another structure is formed up from the ash like a phoenix. As much as individual parts form a single structure, *like brooks make river, rivers run to sea*, so do impurities within a structure break it up the same way as flaws do the percolating cluster. The spread of terrorism from the middle east owes much to the connectivity of air routes.

If we accept Plato's (360 BC)argument that men are the same because all human souls are the same, then any big difference in the cultures of the world can only be satisfactorily explained by using the percolation theory. If all human souls are the same and souls are more important than bodies, then all men are the same and all societies being comprised of equivalent people will necessarily have an identical structure or structural basis. Intrinsic attributes of a structure, for example the culture of a country or the way of life of its population, can take on various forms according to the phase which it has arrived at through the percolation under the various factors which includes historical and geographical influences. One example is the stark contrast on the value given to winning and losing between the western and the eastern cultures. As I see it, for the former the emphasis is on winners, which is reflected in sayings like *winner takes all*, as contrasted

with the emphasis in the latter on losers, for example the thai saying *win and become the devil, lose and become a saint* [bạe pen bra, jàna pen mạr]. As another example, let us consider the japanese saying *akirameta yo nanisama akirameta, akiramerarenu to akirameta*, which are seemingly easy but which has eluded some distinguished western experts in the Japnese language and culture. Months after having resigned from a PhD course in Japan, Keiko Saitōsent me this poem in an email. As a keen archer in kyūdō, that ancient japanese art which uses bows with lengths of 212 to 245 cm long, I think I can grasp the meaning. But not long afterwards, while browsing through the books in a japanese library in Bangkok I came across another translated version of the poem which is completely different to what I think it means. In his translation, Blyth (1959) has written, "I have resigned myself! Resigned yourself to what? I have resigned myself to never being resigned." The reason I say it is different is that my own version is the following soliloquy, "You are beaten! In what way do you say I am beaten? If you said that you shall never be defeated then in fact you have already lost." Kyūdō teaches one that results are nothing. This is not only true for kyūdou., but to all martial arts as well, provided that they are genuine. The training in any art of fighting is never to learn how to fight, but to train the subconscious mind such that it will be able to act by it self at the right time. The subconscious mind is similar to what Freud calls *superego*. In line with Plato's argument that the human souls are never evil, to martial arts this subconscious mind invariably carries the same attributes to that of the superego. In such perspective the Freudian id has no part to play and can be considered as being neither a subconscious mind nor on the same level with the superego; whenever pushed one never push back, and every fight is a self-protective one. The same is the case with kyūdō where there is a difference between an arrow hitting the target and the pride of a successful hit. An arrow hits the target best when it was already there before you took aim. Similar to the idea of complex superposition in the quantum mechanics theory, the free subsconscious mind superimposes and becomes one with the surrounding which includes the target as well as the path towards it through the air. The arrow thus runs home on its own accord. No one could have put it there; certainly not yourself, so there never is anything to be proud of. Only a few months earlier it was that I learnt the art of Kyūdō for the first time when on 14^{th} September 1997 in the 36^{th} Meguro-ku Sport Festival of the Meguro district in Tokyo I led our underdog B Team of three to win second. The other two members of the team were Arakawa Hiroshi and Suzuki Jirō. Looking back I can still remember how when one third through our round and having not scored a single point I suddenly feel one with the surrounding, totally oblivion to the sound of the audience behind us, and let go of the target with the result of which not only that I could break the egg but also the way the manner in which the arrow hit the target was perfect. I am proud to say now that eventhough that hit was the first and only one by me in that event, the other two of my teammates from that point onwards never again failed a single shot through to the end of our round. I wonder if it is not the case that, having been trained to be at ease and relaxed with the bow, my subconscious being, that thing inside me which still do not know much about, kicked in at such the right moment that percolated or communicated itself to my teammates, and then, being sure of having accomplished the task, left the scene. I am rather glad to be left anonymous in this way. To account for my claim of not having consciously aimed that transitional shot, for fifteen years none of my glasses has been adequately prescribed to let me see the bullseye clearly from that distance. Probably nobody will ever know that it was Kit Tiyapan whose name stands first among the three members of the B Team because, reluctant to have my name written in that outlandish katagana appear on three certificates for fear that it disgraces my teammates, I went by the name Zhāng Míng Lóng , what was given to me by my late grandfather and its sole user.

Waking up to the radio on 30^{th} May 2002 morning I heard on the BBC4 one writer as saying that international conflicts and crises occur because matters take on *momentum* which is sometimes difficult to reverse, that it can not be left alone to resolve like playing the russian roulette. I think the only thing that can solve an international, or internal, crisis is a counter momentum built up in the opposite direction. Some says that force stops dangerous mechanisms and that politics never work. No doubt any politics that has no strategies by which to reach at such counter momentum as needed will probably never work.

Life is a dream, or dreams are the other phases of life. If we consider dreams to be in the domains where our minds are in different phases from our awaking selves, then the analysis of the dreams would amount to the study of these various phases. Qualitative sayings like this are sometimes mistakenly written off as nonsensical in the field of science. What is meant when Sacks

(1973), for instance, speaks of 'romantic science' being badly needed is that eventhough art can stand apart from science, science can not be separated from art. I do not think such 'romantic science' is that badly needed, at least not in the field of geometry, mathematics, or percolation. But then again, the first two of these are to me essentially the languages, while the last one the music, and therefore also a kind of language, of nature. Music is the language of the mind and the thing that holds it together. Arts and music heal the mind. But they may not work, which makes them different from medicines. Something mysterious has to trigger before they do. As the *ear* of science they are the things which hold together the *eye* or structure of science.

Opposing forces in nature is an immemorial theme which recurs time and again. This dualism or the law of polarity shows itself, for example, in the eastern philosophy of the opposition between Yin and Yang. This theme came up again in the mid 20^{th} century in the name of percolation, the interplay between two opposing phases. The dynamic triad where the opposing *thesis* and *antithesis* come together and resolve each other into the *synthesis* is analogous with, for instance, the opposition between congested and free-flowing roads in the traffic network which resolve each other in the case where $p_c < 0.5$ into the band of synthesis having the width of $2(p_c - 0.5)$ and centred at 0.5 (§ 5.1).

The conceptualising power of man culminates in the recurrent themes found more prominently in creative minds (Hankins, 1997). These recurring themes represent *faith* in science. And if we accept that every from of faith is ethical, then all these themes are ethical; in other words, the theme of the antagonistic power between Yin and Yang is an ethical part of the eastern philosophy and the theme of the opposition between phases in percolation is an ethical part of the western science.

§ 3.8 Enlightenment

Both discovery and enlightenment are percolative processes. Like a detective gathering clues suddenly understands the crime he has been investigating, past experiences of a person helps him arrive at the solution. It is likely that much more of these experiences is involved in solving even a seemingly easy problem. The spinning or dancing stars in the algorithm I describe for my solution to the continuum percolation of polygons in § 3.8 can be traced back to my experience in the ancient Thai art of fighting using the swords at the Ayudhàyạ school. Here the ray of the star represents the arm of a person and each side of the polygon the sword he holds in his hand. At the school we learnt to always keep the sword at right angle with the lower arm at all times in order not to hit our friends, since this is no real fighting but a practice. Yet because they correspond to circles with greater radii those parts of the sword away from the hands still hits the swords our friends hold in their hands, and in so doing set off beautiful sparks flying which could prove extremely dangerous if it ever gets into your eyes it scares you more than being hit by the sword. The solution for the flexatube puzzle that I discovered while in Budapest in 1990 is another one of the earlier examples in which a discovery percolates throught after a certain amount of thinking about a problem. I think it takes more than those several days it took me, that is much of my previous experiences with other puzzle counts, which makes it possible for me to solve this particular puzzle. This is not to mention about my long forgotten class in Euclidean geometry which I had to study when I was around twelve years old, *etc.*

The Enlightenment movement in Germany was something different. According to Immanuel Kant(1724–1804) enlightenment is to have the courage to think, as he has told us in an essay (Kant, 1784) of his to have the courage to make use of our understanding.† But the idea is much older than the eighteenth century as the *Bhagavad Gītā*, written written in Vedic Sanskrit and being the sixth book of the epic poem *Mahabharata*, says essentially the same thing. Antithesis to the idea of Kant is the refutation of Karl Popper (1902–1994).

† Sapere aude! Hab Mut, dich deines Verstandes zu bedienen! ist also der Wahlspruch der Aufklärung.

§ 3.9 Forest fire fighting

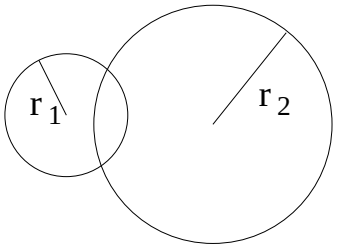
Voronoi graph can be used to aid the decision on the locations of Closed-Circuit Television (CCTV) cameras. CCTVs have proved to be very effective in reducing the number of crimes in Manchester as well as other places within the UK. In order to be effective, it is important to have no large gaps within a control area. Strategically placed, the cameras will give the best cost-effective solution while covering the largest possible area.

Within crucial areas cameras could be positioned in such a way that their ranges of operation intersect without gaps. The points of intersection represent the vertices of the Voronoi regions centred around two or three neighbouring cameras. The design objective is to produce the maximum covered area using the least number of cameras.

Outwards from each crucial area extends space not being covered by cameras. Similarly around other crucial areas surrounding this area also extend such unobserved space. The next step in the design is perhaps to make sure that people walking from one covered area to another are safe. To do this one can consider the whole covered areas as centres of yet another set of Voronoi tessellation orders of unit larger in size. The purpose now is to make sure to position cameras at the positions of vertices of the Voronoi regions of this second set of tessellation.

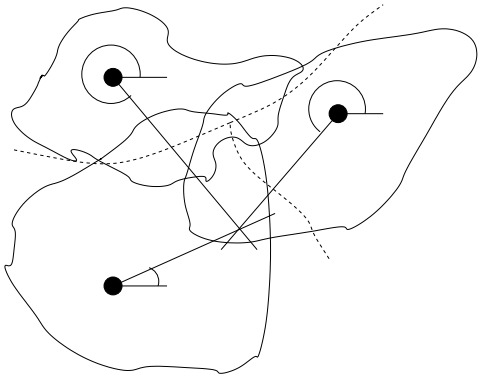
Both the first and the second tessellations will be subjected to geographical constraints. The former would be predominantly influenced by the shape and the position of buildings and surrounding structures, the latter by the paths connecting the covered regions. Here a vertex of these geographically-constrainted Voronoi regions could be a point which is equally far from three covered regions. Another set of positions to be considered for placing CCTVs is one of those points equally far from two covered regions along the shortest path between them.

At a graduate development programme on 16^{th} November 2001 I gave a presentation for ten minutes on a title *division of space*. In it I described the current project of mine as concerning porous media, or division of space in general. The part presented is from a section for miscellaneous applications. Mathematical models of partitionings, for example the Voronoi tessellation pioneered by G. F. Voronoi and G. L. Dirichlet in the 19^{th} century, help towards understanding physical Euclidean and non-Euclidean world. Procedures based on these models help solve many physical and strategical problems. When I was fourteen years of age I was trained in a course for those who volunteer to fight forest fires in the area around the Sudęb mountain, close to where I used to live. Although after the course I never had an opportunity to use it, the knowledge obtained from that course has given me interest in forest fire fighting. To mention but one of the relevant topics contained in that course, one can imagine a forest fire station as being a nucleus of a Voronoi partition of area under the protection of that stations. Here the distance from a station to any point on the forest ground is not a straight line but the time it takes a fire fighter from that station to reach the point in question. This time is affected by the *tortuosity* of the path, in other words the degree of winding, as well as the difficulty of the climb. There normally is an observation tower at each fire station, so we may safely assume that every nucleus is equipped with one. Then the exact location of a fire observed can be located on the map from the intersection of lines of sight drawn from the neighbouring towers towards the direction of the fire. These lines of sight only give the position of the fire on the map. The distance measured along such a line does not necessarily correspond with the Voronoi distance which has a unit of time. The area covered by a tower depends on its height; a higher tower can report a fire occurring at a further distance. The area of covering is circular if the forest is a flat plain. But such a case is rare and in general this area is a distorted circle the degree of distortion of which depends directly on the contour of the land around it.



In Figure ℵ if h is the height of a tower and r its radius, then $r_1 < r_2$ implies $h_2 > h_1$.

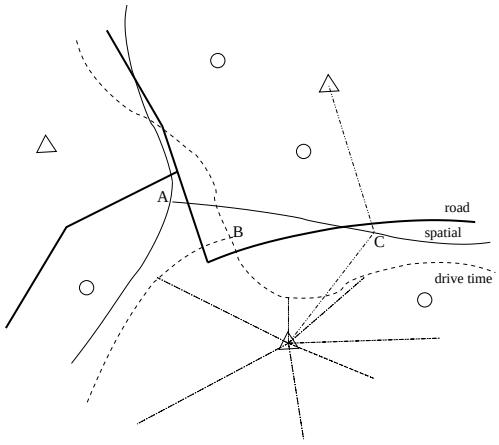
ftower. Range of observatory towers.



In Figure ℵ the Voronoi boundaries are not straight lines as a result of tortuosity. Observations from only two towers suffice for finding location of the fire, but an observation from the third tower would reduce the probability of an error.

ftuos. The dashed lines are boundaries of Voronoi regions, solid boundaries the observation range of the towers, and straight lines from nuclei the lines of sight.

I also talked about strategic locations. I once listened to a job presentation by Tesco where I learnt that they use the Geographical Information System to place their new stores in strategical locations. The planning department at Tesco uses what they call the *porcupine diagram* which looks like the rays of light emanating from a star or the quills from a porcupine. These rays radiate from a supermarket and their length represents the drive times required to get to the store from the other end of the line. Factors affecting the locating of new stores include for example drive time, competition, logistic, parking, and traffic.



The spatial boundary of the Voronoi diagram is in general different from its drive time boundary, as shown in Figure ℵ where broken boundary lines represent Voronoi partitions when the distance is the drive time while solid boundary lines that when the distance is the spatial distance.

fstra. The porcupine diatram superimposed on a map. Triangles are Tesco supermarket stores while circles are those belonging to their competitors.

A similar approach to the above can be applied to help locate the Closed Circuit TeleVision cameras in a strattegic location. Suppose that now the triangles in Figure ℵ are important places, for examle business centres, schools, or universities; let us call these *sensitive areas*. Then security cameras should be placed at both points A and B as well as at the sensitive places. Here the solid boundary lines meeting at the point A represent the Voronoi boundary for walk time instead of the actual spatial distance. Moreover, a camera should be placed at the point C which is the point where the shortest walkway linking two neighbouring sensitive areas meets with the Voronoi boundary of the walk time, in other words a point which lies midway along the shortest path that joins nextdoor sensitive areas. The reason for this is that these points are likely to be frequented by people going from one sensitive area to another, therefore they are prospectous for robberies. On top of that if a robber wants to flee from one sensitive area to another his sensible choice would pass through C while A with its connecting three different routes would be a clever choice to choose iconfound a pursuer, or likewise B if he is driving.

I had been first to the police headquarter and then to the city council office because I wanted to know whether security cameras have been placed at locations analogous to these points A, B, and C, but has been told that the information I asked for is a sensitive one. The answer is understandable because at the time President Bush was trying to get his hand on Bin Laden and Prime minister Blair had just declared Britain at war and alert. However this was not good for my theory since I had no way of knowing whether this approach is a new one, or whether it could be used to improve the security in the city even further. I hope that most of these strategic locations A, B and C have already been covered by security cameras even if the decision to do so had been arrived at by some other theories different from the one I suggest. Unlike in the case of sensitive places, the cost of the cameras to be placed at these strategic places may have to be the responsibility of the city council because their importance is not immediately obvious owing to their locations. In contrast, most schools, universities or businesses should be willing to help with the cost of such installation from which they would directly benefit.

On one slide I explain how one can divide transformations into affine and non-affine ones. The affine transformations are governed by the equations $x' = ax + by + p$ and $y' = cx + dy + q$. As a conclusion, mathematical models are beautiful because they give us understanding, and they are useful because they stem ideas and prpocedures necessary for solving problems. The presentation has gone from forest fire fighting to crime prevention without my knowing it, so the future works are to collect real data and practices, to analyse and compare with the theory and simulations, and then to implement and monitor the operations which I think unlikely to happen because it is impossible for me to become a member of the Greater Manchester police force. Dr. Jim Boran who was present

gave *good*'s for project understanding, overview and structure, summary, examples, use of colour, connection with drawings, enthusiasm, pointing at screen, and familiarity with the material. He wrote that the explanations were clear, the content technical but with good personal view, but that I looked at the screen too often and at times was difficult to hear. Bigger lables are suggested.

Naval forts are positioned with an idea similar to forest fire stations. Each fort forms a circle of effective defence around itself. These circles form a chain like beads on a necklace. One example (*cf* Petersburg, 2001) of this are the forts of Kronstadt near Kotlin Island in the Finnish Gulf, 20 kilometres from Saint Petersburg, which have been used as a base for the Baltic Fleet and is now controlled by the Russian Navy. Here seventeen forts had been built between 1704 and 1896 which form a guarding barrier for the city of Saint Petersburg. Examples of these forts are the Totleben Fort, founded 1886–1913, and the Alexander I fort, founded 1836.

In the study of traffic in § 5.1 we consider the percolation of roads, which are edges of the structure. We can also use the same structure for the study of fire protection and evacuation in cities, in cases of disaster, but now it is instead the percolation of cells that interests us. Roads form effective fire barriers as well as divide the conflagration into partitions. By carefully managing these, for example putting more efforts on the key or critical partitions, the flames can be contained more quickly and the damage minimised. These zones are also important for evacuation planning, for example that made by the Greater Manchester Police. In Tokyo where there is always a fear of big earthquake, informations are given out in plenty to tell people what to do in case such a disaster as the Great Kanto Earthquakewhich occurred on 1^{st} September 1923 should struck. School grounds are usually assigned as evacuation centres. And as the distance one has to walk to get to one of these is crucial for one's safety, the evacuation zones assigned by the city councils there naturally divide the city into domains that approximate the Voronoi tessellation. In 1998 when I was in Tokyo the fear reached its height, since it was generally believed that big earthquakes in Tokyo occur at a period of approximately seventy-five years.

§ 3.10 Fractals and percolation

In doing simulations on filtering membranes, one usually assume that the blocking particles are all of the same size. Or one may apply more intuition and experience, and say that particles should have a normal distribution. Any mineral engineer will be able to tell that it is normally the case that the distribution of aggregates according to their sizes is normally normal. But one must keep in mind that the normal distribution here is by weight not number of particles. Therefore it is unlikely, except in a few special cases, that simulations which assume a normal number distribution of blocking particles will represent the real process.

A pile of aggregates with a normal distribution is likely to have a fractal number distribution (*cf* Liebovitch and Scheurle, 2000). Such fractal distribution represents a hyperbolic graph, which on a log-log scale is a straight line having a negative slope.

So far we have only considered percolation by competition between two phases. We have looked at networks in a cubic box and have assumed that the network is initially available in its entirety. But what if one has a competition of three phases or more? Or what happens when there are only parts of the network from the start? The picture of this can not be clearer than that of politicians arguing for votes, or the opinion polls of people across the social cross-section. Lustick and Miodownik (2000) consider this problem in the context of politics where the agreement clusters necessarily appear in various colours not merely black and white.

Applying this to percolation and a whole new dimension opens up for investigation. One possible approach is to consider only an arbitrary part of the network within our cubic box. Clearly this is only meaningful if the part being considered is itself a percolating cluster. Carry this one step further and we can have percolation within percolation, when each cluster within the network is itself subject to another percolative process, threatening to destroy it for instance. Such a dynamic scenario is a yet untread water which should open up much ground for investigation.

Porous media

§ 4. Porous media

§ 4.1 Medical science

Many diseases can be best thought of as related to percolation when a threshold exists passing the point of which it is difficult to return or to get back to normal. For example a cut may take several weeks to completely heal, but muscular strains could take several years and even then they usually leave a weak spot and thus can not be considered as completely healed. This means that despite the turn over of cells within the fibers of the muscle, the strain still prevail. In other words it has buried itself into the structure of the fiber and no longer is related merely with the individual cells. Not only muscular fibers, but also the brain and the nervous system may be viewed as a kind of networks. Therefore it is not difficult to see many nervous or brain diseases as percolated phenomena. Madness and its onset is an easy example. Deafness, either gradual or sudden, is another one. The difference between a sudden and a gradual onsets of deafness lies in the position of the percolation threshold on the timescale. In the former the network percolates after a period of hidden graduation or deterioration in the condition of the inbuilt sound detector, while in the latter it percolates before the graduation down towards the deafness. Most books on otolgy talk about sudden onsets of deafness, normally overnight. The reason they are much talked about is more likely to be because of their tragic nature than because they are common. It is interesting to note that the word *otology* means the study of the ears, *-logy* is suffix which comes from the Greek word *logos,* meaning *the study of*, while *ot* or *ōt* means *ear*. It is probably a coincidence rather than a liguistic relationship between the two languages that *oto* is the Japanese word for *sound*.

I should have something to say about the Menière disease because both my previous general physician Dr. Sreedharan and the present one Dr. Chan think that it is what I have. I do not quite agree with them, but if both of them were right and I am wrong then Menière disease results from the accumulation of certain things which then percolates, and I think I know what these things are. This would have been rather a medical breakthrough because all medical textbooks that I have come across anonymously agree that this is a disorder the cause of which is excluded from its definition. In other words all we know about is what its symptoms are, nobody knows what causes them. What I maintain is this; provided that both of my doctors diagnosed correctly then the cause of it must be percolation. Having said that, I do not think that they did diagnose right, for eventhough I have both the pressure in the ear and the tinnitus, I never have vertigo. It is true that I have some experience of syncope which is even worse than vertigo, but they are not the same thing. Physicians, however, insist that you must have vertigo no matter if you say you don't, because then they would know that you have the thing called Menière disease the definition of which they do not know anyway! The disease is named after a french physician Prosper Menière who first described it in 1861.

Tinnitus is the constant rumbling or ringing in the ears. The sound can be real and sometimes can be heard by others. Many things can cause it, for example (*cf* Andrews, 1997) excess noise level, alcohol, allergies, antibiotics, blood glucose or blood pressure swing, circulation changes, cranial or jaw-joint misalignment, fever, inflammation, temporomandibular jaw joint diseases, tumours on acoustic nerve. It can also be caused by metallic – aluminium, lead, or mercury – poison. If the cause of it is a prolonged or excessive level of sound, then the process would be similar to that which causes, for instance, the repetitive strain injury.

I will now explain why it should have something to do with percolation. First, I think it is sound or the excess of it which causes the percolation. Next let me give some facts which lead me to the conclusion above. Around the turn of this millenium a friend of mine bought two concert tickets and invited me to join her. Seldom been to a concert, I was delighted. And having noticed that the program includes a piano recital of one movement from a sonata, probably the fifth, by Scriabin, I decided that I would do my homework in order to gain as much as possible from the concert. At that time I had recently bought the music, a complete sonatas by Alexander Scriabin. They are a total of ten sonatas, and I sat at the piano and read them one by one, day after day. Ten hours a day at the piano was sufficient to give me the obvious sign of an ear problem. For a few days I could not hear anything clearly, and could hardly speak normally. The second sign came not long after I had been to a Community Action conference held in Liverpool. The conference was something new and was very interesting, but I discovered there that young volunteers enjoy too loud a music on the dance floor at night to the cost of my ears. Again the following few days gave me the most alarming feeling, I hardly could hear anything, and whatever sound I heard seemed like coming from the outside to me inside a sealed tank. This together with the experience of having lived in Bangkok, the city where noise polution ranks number one, and that of having listened to loud music

myself when I was young, have combined to reach the limitting threshold. As I am writing now I still have yet to recover from the second trauma to the ears mentioned. There is a saying that the third time pays for all (*cf* Tolkien, 1955), and so I hope that the third time shall never come considering that the second one has already been this bad and I am in doubt whether I will be able to recover from it. But to me it is clear that this has got something to do with a limitting threshold and thus percolation.

In the case of Beethoven, who also had an ear problem which results in a total deafness, most modern physicians think that the cause was multiple sclerosis, which I also disagree. As a great pianist and a conductor he was exposed to much sound, so I suspect that the latter is also the culprit in his case. It seems an irony that for such people, as well as for myself, music is life and music hurts.

Some thinks of multiple personalities as the fragmentation of the mind or brain (*cf* Keyes, 1999) which causes a person to identify himself as different personalities each of which has its own memory separate from the others. There is a transfer of control at the switching from one personality to another, and all the personalities could use the same body in turn. It is more likely that instead of fragmentation the various personalities are different phases coexisting in the same network of the mind. Like a polyglot or a normal person switching phases among different languages he knows, a person with multiple personalities unconsciously switches himself from one self of his to another. Multiple personalities often report head voices of one self talking to another. These head voices are distinct conversations and thus are different from those experienced by people with schizophrenia. The various personalities have different Electro-Encephalograms, Intelligent Quotients, and psychological test results. This altered state phenomena is normally cured by fusing the personalities together, but some say that the this is not necessary as long as all the personalities are aware of one another. Similar to multiple personalities a person is often said to have good and bad sides, or in some cases selves (*cf* Stevenson, 1886).

We still do not understand much about sleep, that state where the mind finds itself in phases which are different from the awaking self. No one seems to know why it is neccessary that we sleep at all, but all animals sleep, or die if they are deprived of it. Dolphins sleep one hemisphere of the brain at a time. Sleep and wakefulness are two different phases of the mind, and the sleep itself may contain more than one phase. The transitions back and forth among these various phases are the same as the changes of phases characteristic to percolation. A person normally has several dreams while he sleeps, but only the immediate one from which he is wakened up is remembered. This may mean that during sleep the mind is switched among various phases, or it may mean that in sleep the mind only has a short-term memory.

Subconscious mind can sometimes come to the front. This often happens when the conscious mind becomes weak. When a man becomes drunk, for instance, he may be able to find his way home only to wake up the following morning, sober and wondering how he has got there. This is because in the drunken state his mind becomes muddled up by the effect of alcohol that the subconscious mind took control to lead him home unconsciously. Multiple personality represents the fragmentation of the subconscious mind into virtual clusters each of which then alternately comes to the front to parade as a person. These clusters are probably superimposed in three or higher dimensions, without overlapping one another, in a way similar to the intertwining of clusters in a three-dimensional Voronoi network. The change of personality in this case is then a transfer from one phase to another. The first and most important fragmentation which leads towards a multiple personality seems to happen only in children at around the age of three, when the mind is in the forming and the language of the mind is predominantly geometrical, while later and minor fragmentations are possible much later afterward. Our minds during sleep seem to be another layer separate from and exists below both the conscious and all fragments of the subconscious mind, as can be seen by the fact that a person with multiple personalities also needs to sleep. Some of the most fearsome and the least understood sicknesses like Parkinsonism and Encephalitis Lethargica, or the sleeping sickness, affects only the conscious part of the mind, not the higher faculties of the unconscious part (*cf* Sacks, 1973). One of the effects of the medicine laevo-dihydroxyphenylalanine, or L-DOPA, seems to be that of letting the subconscious mind take over to motor functions. The condition of the patients usually improves quickly soon after first put on the medicine, then overshoots a stable condition and shows adverse effects of the medication which are no less gruesome than the symptoms before starting to take the drug. These counter symptoms probably occurs in Control Systems as an overshoot in the output in response to a step function which represents the usual constant daily dosage used. Because such undesirable symptom occurs in most cases, dosage should

either be reduced before the overshoot occurs in accordance with the empirical time constant of the response, or increased very gradually from zero at the beginning in order to reach the steady state be reached without an overshoot, as an analogous to an over- or critically damped control scheme. One of the most intriguing effects of too much use of L-DOPA is that primitive, sometimes subhuman, behaviours could be brought to the front while the patient himself can only stand back as a witness to his own self without being able to do anything to stop them.

In the art and science of Neurology there is no lack of phase transition phenomena. So much so that symptoms normally come in pairs of opposing poles, for example aboulia – hyperboulia, automatism – command-negativism, bulimia – anorexia, and perseveration – block. Within these pairs the poles may switches suddenly as can be seen in kinexia paradoxa, the explosive transition between hyperkinesia and akinesia. Somewhere between the two poles lies our normal conditions. In many of post-encephalitic patients taking L-DOPA, this middle ground which starts off very wide can rapidly shrink until the patients find themselves balancing on the edge of a knife and it becomes impossible to calibrate the drug to the right amount.

§ 4.2 Zeolites

Zeolite is the name given to a group of minerals with a porous structure. It is used as molecular sieves as well as in many chemical engineering processes. It may be an important ingredient responsible in lubricating the fault lines in earth crusts, thus making earthquakes less severe. Evan *et al* (1995) found foliated cataclasite and ultracataclasite in the San Andreas faults. These are composed of clay and zeolite. In the case of the ultracataclasite, there are fragments of 20–100 μm diameter feldspar and quartz embedded in a matrix of clay and zeolite which has grains of sizes smaller than 10 μm.

The chemical composition of zeolites is that of sand, *i.e.* aluminosilicate. Their general formula is (Dana and Dana, 1997) $(\mathrm{Na, Ca, Ba})_{(1-2)}(\mathrm{Al, Si})_5\mathrm{O}_{10} \cdot \mathrm{nH_2O}$. They differ from sand in that they have large internal cavities. These cavities increase their internal surface area as a result of which they can be used as catalyst or molecular sieves with both shape- and size-selectivity. One example is the use of zeolite as sieve to separate iso octane, which has high antiknock property, and iso pentane from octane and pentane. Zeolites can not withstand high temperature because they rapidly lose water. Eventhough this process is reversible, their molecules collapse at temperature higher than 600°C. Around 41 types of zeolite occur in nature. Linde Division of the Union Carbide Corporation produced the first synthetic zeolites in 1950s.

§ 4.3 Fluid flow within networks

Flow of viscous fluids through networks of geometrical objects is an important topic in various disciplines of engineering. Happel (1959) studies two cases of the flow of viscous fluid relative to arrays of cylinders, one parallel while the other perpendicular to the cylinders. This is found in practice as the flow through a bundle of heat exchanger tubes.

§ 4.4 Material science

A chiral is a group of points, or geometrical figure, whose mirror image can not be brought to coincide with itself. Achiral is the antonym of chiral. A chiral object is an object which fails to be achiral. Chirality is a purely geometrical property since since all the operations involved, namely the plane symmetry of the reflection, the rotation, and the translation, are all isometries. Chiral is used in the studies of molecules and knots.

The Poisson's ratio, ν, is the ratio between the transverse contraction strain and the longitudinal extension strain, that is $\nu = -\frac{\varepsilon_t}{\varepsilon_l}$. The theoretical value of the Poisson's ratio can range from -1 to 0.5, but for normal materials it is generally positive. At $\nu = 0.5$ the bulk modulus is much greater than the shear modulus and the material is incompressible, while at $\nu = -1$ the opposite is true and the material is very tough and highly compressible. The bulk modulus B, and the shear modulus G are relatated to each other by the equation $B = \frac{2G(1+\nu)}{1-2\nu}$. For rubber this value is 0.5, for aluminium 0.33, while for cork it is approximately zero. Materials with a negative Poisson's ratio have been found whose structure is re-entrant (Lakes, 1987). Applications of such materials include robust shock absorbing material, fasteners, and stoppers of the wine bottles.

§ **4.5 Forces between particles**

Essential to the simulation of stochastic models of particles is the consideration of forces acting on each particle (*cf* Schumacher, 1996). Since in theory the forces acting between two particles extend their effects to infinity, one has to make approximations. The degree of justification to these aproximations depends on how the force in question vary with distance.

In the case of the Coulomb force where the potential energy is $u(r) = q_1 q_2/(4\pi\epsilon_0\epsilon_r r)$ and the force $f(r) = -\mathrm{d}u(r)/\mathrm{d}r$ is $f(r) = q_1 q_2/(4\pi\epsilon_0\epsilon_r r^2)$, the justification is high as is seen in Figure 4.1 where the force-distance is a decreasing straight line on the log-log graph, q_1 and q_2 are protons, and r the distance in metre.

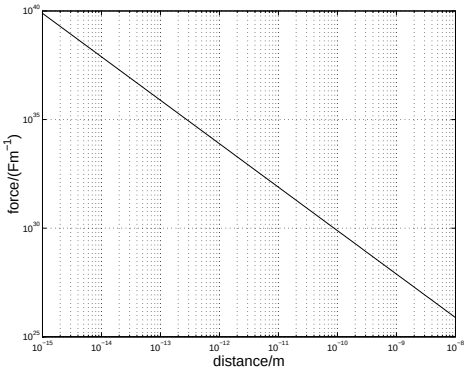


Figure 4.1 Coulomb force *vs* distance in log-log scale.

Application of percolation in traffic modelling

§ 5. Application of percolation in traffic modelling

§ 5.1 Traffic networks

It does not take much imagination for an average person to see that traffic congestion is a percolative process. As I have lived and studied in Bangkok where, at that time, the traffic jams were renowned as being second to nowhere in the world it would have been peculiar indeed if after having acquainted myself with the percolation theory I do not apply it to the study of traffic systems. I remember once in 1990 it took me more than five hours on a bus to travel the distance of ten kilometers or so along the Sukhumvit Road in Bangkok. Being then in my last year at the university and having missed an important meeting with my supervisor that day as a result, the reason I gave to my supervisor must have sounded like a lie. Only the newspaper of the following day could save my integrity. In it there was a report that the road on which I had spent a whole day travelling had been given a footage on the BBC World news.

Soon after having decided to investigate the application of percolation to traffic, I gave a poster presentation at the Fourth Annual Conference of Thai Researchers in Japan in 1997, the proceedings of the event of which also contain a number of abstracts from some of the researches that I was currently working on (Tiyapan, 1997). The poster presentation was under the title *Fractals in the modelling of traffic networks* and was reasonably a success in that it interested several people there and I was asked a number of interesting questions. One of the two honourary chairs of the event suggested that I wrote a detailed article for the monthly journal Sakkayaphab (*Śakàyàbhạb*) which is published by the Association of Thai Professional in Japan (ATPIJ), but such a paper never materialised as I found it difficult to write about the topic in Thai and I was reluctant to have my article translated from English again because there have been mistakes in earlier such attempts and the editing team does not always bother to tell the reader that the articles are translation (*cf* Tiyapan, 1996). One mistake which stands out is the translation of a passage in my manuscript,'The irony in this is that while the money involved increases exponentially in relation to the cases of its counterpart which are known as the *blue-collar* crimes, the risk of being caught is comparatively small', into a translated version which means the opposite, 'Naịkhảṇadị̀ ạjàkamr [sic]‡ ịk prabhẹd gụe *Blue-collar crimes␣ zụ̀eng bọaban kaḅ ngoen caṃnoan mạk. Mụ̀a prịaḅdịaḅ kan lậew ạjàkaur* [sic]¶ *lhăonī̀ thụk caḅkum mạk kvậ ḅọaeḅ rạek.' [While another type of crimes are Blue-collar crimes which involves a lot of money. Comparatively these crimes are caught in greater number than the first type.* Whether I could write good Thai or not, from this lesson I think that from now on I should translate my own articles.

In fact, it was not so much a poor translation of my article that I mind but the failure of the editor to acknowledge that the article was a translation. As it is the case, whether this be coincidence or not, translation quality tends to be better when the name of the translator is prominently written (*cf* Tiyapan, 1996). In the reference above, only minor mistakes appear in the translations, namely 'however' translated into a word which means 'moreover' (*yang*) and similarly 'metaphor' into that which means 'simile' (*upàmạ*).

The paper in § ℵ, except from a few minor spelling corrections, is an exact reproduction of the paper I submitted to the American Journal of Physics in Nomvember 1997. I feel that these minor changes are justified if only for the reason that the paper has not been accepted for publication. I include it here in order to show the development of ideas leading up to the present work. Towards that end all the contents, and as far as possible even all the syntactic and grammatical mistakes, have been left untouched. Some of the figures have been left out due to the loss of my home directory on a computer in Japan which I have learnt about soon after my arrival to resume my Ph.D. research in Manchester. The address of correspondence listed is my home address eventhough I was then with the Tokyo Institute of Technology. This was only to avoid further misunderstanding in cases where letters to me had been opened before they reach my hand without due apology. I suppose that the idea about the unity of a laboratory there amounts to a different standard of privacy and respect from what I have been brought up to believe in. There, for example, the circulation of preprints received for refereeing among the nearly twenty members of the lab is a matter of course.

The supposedly inferior quality or contrivedness of the paper can probably be ascribed to various factors, for example the constant dread of always being observed (though I have no doubt whatsoever about the relevancy of such topic to modelling in control systems), constant and to me

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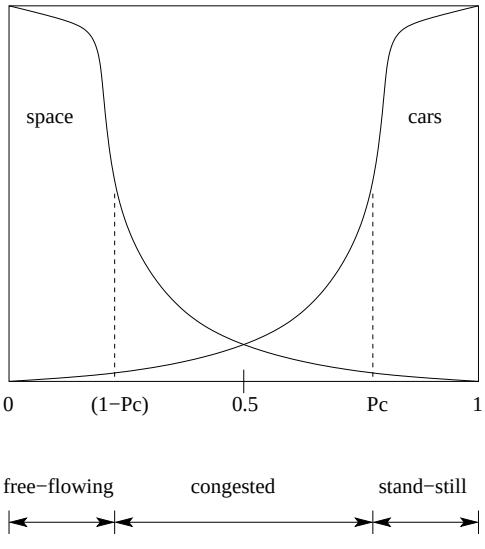
‡ should be *ajàyạkamr*

¶ should be *ajàyạkaur*

somewhat purposeless drive towards publications by all possible means, and last but not least my own shortcomings. Nevertheless, I still believe that the approach I proposed there is sound and the idea new.

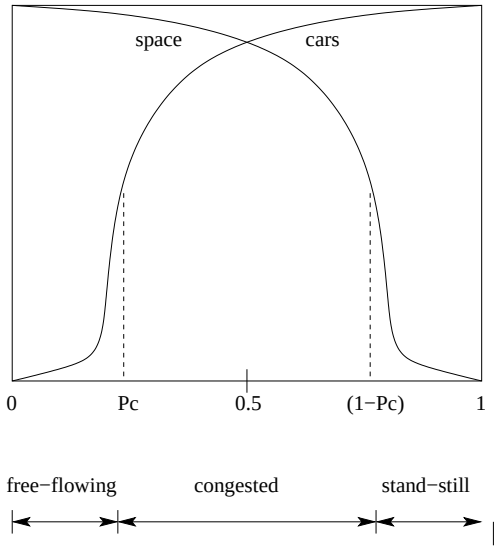
My interest in traffic began at the age of thirteen when I as a boy scout volunteered to become a traffic scout to help managing the traffic, mainly around my school but also occasionally at some other places around the city of Chiangmai, Thailand. The work, which I had done for two years, has given me a first hand experience but is not directly related to the following modelling of traffical network.

In the simulation the networks are drawn which have their vertices as points where two or more roads meet one another. The proposed study is to compare the robustness of two traffical networks by comparing their percolation thresholds. When a new motorway is planned, for example a ring road around a city, the two networks, one with the ring road and the other one without, can be simulated to find their percolation thresholds and then these values compared.



The traffic condition is generally described as free-flowing, congested, or stand-still. For cases where $p_c > 0.5$ as the one shown in Figure 5.1 the stand-still traffic corresponds to the situation where only cars have percolated but not the space, that is the places on the road available and accessible to the cars. A free-flowing traffic is where only the space but not the cars has percolated, and a congested traffic is that when neither of the two has.

Figure 5.1 Traffic status or condition when the percolation probability p_c of cars is more than 0.5.



On the other hand, in the case where $p_c < 0.5$ shown in Figure 5.2 the critical probability of cars is now on the left hand side of the middle line instead of on the right hand side thereof in the previous case where $p_c > 0.5$. Definitions of the free-flowing and the stand-still statuses remain the same, but the congested traffic is now the traffic where both cars and space have percolated. An interesting question is whether or not there is a difference between the congested area in Figure 5.2 and the one in Figure 5.2.

Figure 5.2 Traffic status or condition when p_c is less than 0.5.

A *modelling algorithm* which studies the percolation of a traffical networks starts by finding all the vertices and edges forming a network from the road data. Then a blocking algorithm operates by randomly shutting off one edge after another until the network percolates when the critical probability of the network may be calculated. A *control algorithm* for the real time traffic control, however, is as shown in Figure 5.3.

Let C means *cars have percolated* and S means *space has percolated*, then in Figure 5.3 the cases I, II, and III are respectively $\neg C \wedge S$, $(\neg C \wedge \neg S) \vee (C \wedge S)$, and $(C \wedge \neg S)$. Case I is the normal congestion, nothing to worry about. Examples of the control schemes used in Case II are overriding of the traffic lights by the manual control of traffic polices at certain strategic points, temporary one-way systems, bird-eye view observation and feedback from helicopters, traffic control centre, distributed control centres together with traffic radio channel broadcast. And lastly the most important and critical Case III the emergency plan of which may include directing all cars away from congested clusters or if necessary out from the city, and directing all incoming traffic such that no more cars may enter the city until the emergency status ends.

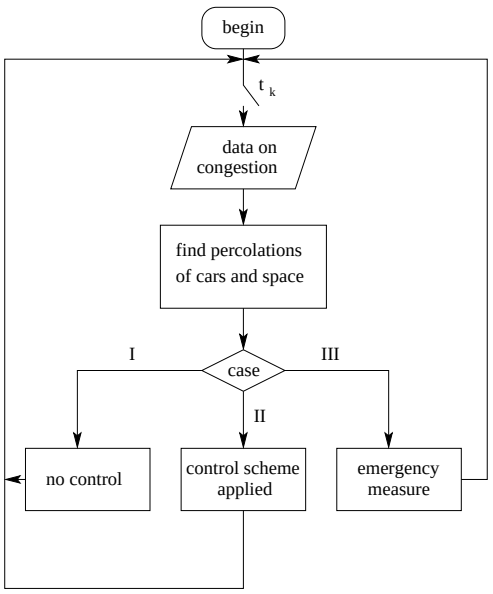


Figure 5.3 Proposed traffic control in real time.

A congested cluster can be broken up by forming a one way flow channel cutting through it

which leads cars away from the cluster. How far the channel needs to go before letting the cars on it circle and seep back into town depends on how badly congested the traffic is. Although it is a normal practice to lead cars along a long detour because this brings in more road surface and thus enlarges the network, identifying the percolationg cluster and cutting it into two or more parts by guarded flow channels is much less common or even unheard of. I think the latter is more important and will lead to a better and more effective control, namely the control fo the percolating, in other words the biggest, cluster.

The percolation probability is important for the networks of traffic both inside cities and among them. It shows the degree of connectivity of the area being considered. Urban road networks have a general character which differ from one country to another, the simplest construction of which seems to be that of the square lattice. One can find this theme of the square lattice and its variations, similar to the dislocations and defects found in minerals, in America. Examples are Denver, Aspen, Durango, Pueblo, Salida and La Junta in Colorado; Boise, Pocatello and Twin Falls in Idaho; Butte, Bozeman, Coeur d'Alene, Kalispell in Montana; and Cheyenne, Laramie and Sheridan in Wyoming (*cf* Florence *et al*). The Great Junction in Colorado and the Great Falls in Montana are very close to being perfect square lattices. As more examples of these (*cf* Collins USA, 1999), in Arizona there are Phoenix, Yuma, Tucson; in California Bakersfield, Central San Francisco, Central Sandiego, Fresno, (Central) Los Angeles and vicinity, Modesto, Sacramento; in Colorado Fort Collins, Denver and vicinity, Greeley; in Florida Central Miami; in Georgia Central Atlanta; in Illinois Champaign and Urbana, (Central) Chicago and vicinity,, Quad Cities, Rockford; in Indiana Fort Wayne and Indianapolis; in Kansas Topeka and Wichita; in Louisiana Central New Orleans; in Maryland Central Baltimore; in Minnesota Central Minneapolis and Central St. Paul; in Missouri Central Kansas City; in Nebraska Lincoln; in Nevada Las Vegas; in New York Manhattan; in Oklahoma Lawton, Norman, Oklahoma City and Tulsa; in Pennsylvania Central Philadelphia; in South Dakota Sioux Falls; in Amarillo, Central Houston and Lubbock; in Utah Central Salt Lake City; in Washington Central Washington D. C. and Central Seattle; and in Wisconsin there is Central Milwaukee. One example in Canada is Toronto. The square lattices of these cities are sometimes cut through by motorways or interstate highways as is the case in Amarillo, Texas. Or they can be surrounded by a ring road or a county highway as is what happens with Lubock, also in Texas.

When the percolation probability is greater than 0.5, we have the interval $p_c \pm (p_c - 0.5)$ where neither the blocked nor the free roads percolate. If this interval is narrow, that is if $p_c - 0.5$ is small, then within this interval the condition of the traffic is very sensitive, and even a seemingly small change may lead to a standstill or instead to a free-flowing traffic. This is easily visualised, since in such situation there would be small islands of free-flowing roads within a large congested cluster, and vice versa small clusters of congested roads within an otherwise noncongested area. Despite their sizes, such small islands of anomaly in either of the phases are particularly important. Moreover, their importance increases the closer p_c is to 0.5. The same characteristic happens in politics where, in the case of coalition governments, a minority party which has relatively few representatives can become critically important and influential to the major party when the latter needs them in order to be able to govern (Ireland, 2002).

Plan to build a ring road around a citie usually includes flyovers, overbridges, and tunnels in order to avoid having intersections. The design philosophy used in Europe is to have heavier traffic goes under a lighter one, which results in either the ring road going into a tunnel or having overbridges or viaducts over it. The philosophy used in Thailand which used to be for the heavier traffic go over the lighter traffic, which puts a more severe limit on the weight of trucks and is therefore not economic in the long run, but this has started to changed, if only to follow the practice of the west. Ring roads do not necessarily resemble a circle, as in the case of the circular speedway proposed around Saint Petersburg (Petersburg, 2001), which is in the shape of a cashew nut. It will link the arterial roads of the city to Helsinki, Kiev, Moscow, Murmansk, and Tallinn into a network. The route proposed is 155 kilometres in length, has 31 bridges, 16 overbridges, 55 viaducts, and will support the volume of traffic of 21 million tonnes. The implementation and contract work is looked after by a joint-stock company KAD Sankt-Peterburga, under the order of St. Petersburg and Leningradskaya Oblast.

One classical example of a ring road is the M25 motorway which forms a circumscribed ring around London. Another more recent example is the M60 orbital motorway around Manchester. In the case of M60, various sections of existing motorways have been put together and renumbered. The northwest quarter used to be M62, the southwest one M63, and parts of the remaining used to be M66. The motorway forms a complete ring around Manchester since 2000 with the opening

of the final northeastern part which stretches from Denton to Prestwich. There is another smaller ring, an inner ring, which is formed by the A6010 and A576. Orbital motorways around cities have now become indispensable and are the hallmark of a city.

6
Percolation theory to economic models

§ 6. Percolation theory to economic models

§ 6.1 Object-location

The object location problem is geometrical in nature. My design exercise (Tiyapan, 1994) during my MSc course, supervised by Dr. Zarrop of the Control Systems Centre here at UMIST, is included here in § ℵ. Figure 15 to 17 which replace the original ones are produced by the original respective programs with minor correction, namely in the drawing of the lower two sub-figures in each one of them. In addition, Figure ℵ and ℵ give an idea where the mask is relative to the object at each instance of time.

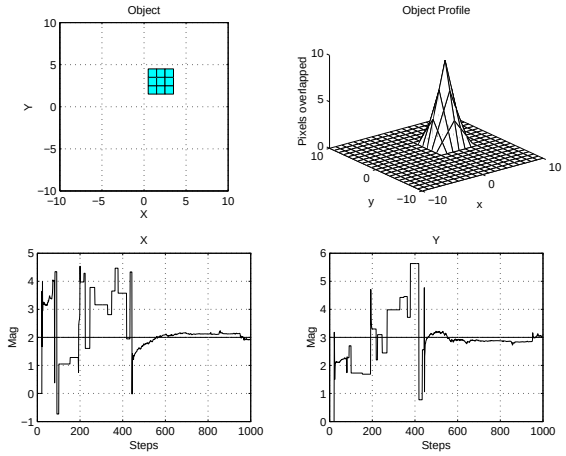
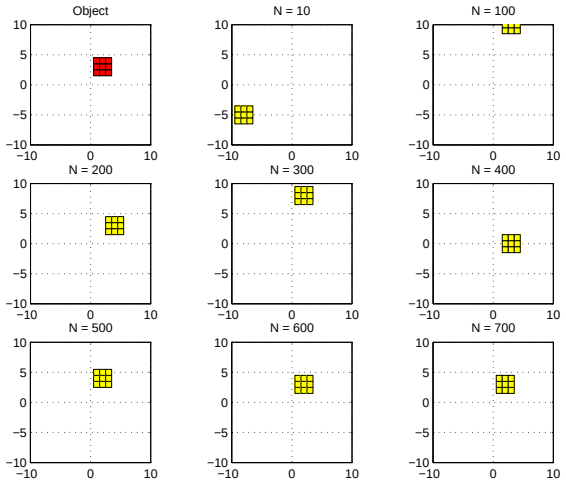


Figure 6.1 uses the same program which produces Figure 15 of § ℵ. It shows the square object, the hill derived from the area of intersection with respect to the position of the mask, and the x- and y-axes of the mask with respect to time.

Figure 6.1 Object location of a square object by a square mask of equal size.



While doing a random searching the program may at times think that it has found the object somewhere and jumps to that location, only to realise later the mistake. This is due to the interference by the noise. On the other hand if it knows that it is doing the random searching, it retains the latest position that it thinks is the correct one. All of this is shown in Figure 6.2 and 6.2.

Figure 6.2 The object together with eight snapshots of the mask at different time.

The paper reproduced in § ℵ has been submitted to be considered for presentation at the 7th International Symposium on Dynamic Games and Applications which was to be held in Japan on 16–18 December 1996. In April 1996 I submitted an extended abstract of the same to the organising committee and it was accepted. I thereby emailed to Dr. Zarrop to ask whether he would mind being a coauthor of the finished paper since it was him who gave me advises on the work which was the product of the design exercise course that I did with him, he said he would be happy to be one and I wrote to thank him. But upon seeing the finished work he wrote to tell me to be its sole author instead because he sees nothing new in it and, without giving suggestions on how it could be improved, rather have nothing to do with it. I duly sent the manuscript with his name taken out in July and it was not accepted.

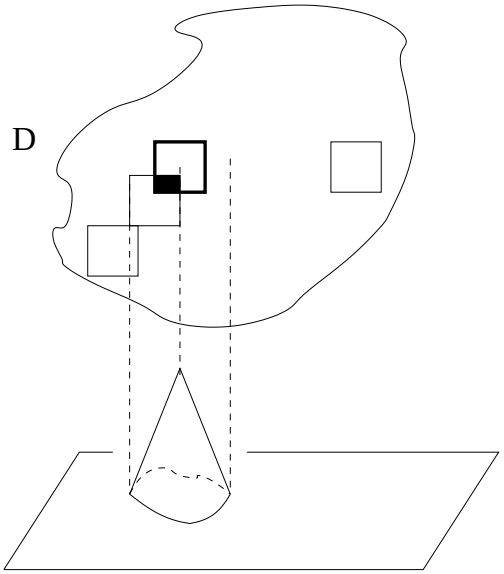
A work by Tiyapan (1996) on object-location problem is reproduced in § ℵ because the original manuscript and results is irretrievably lost. This appears in the proceedings of the ATACS-96 conference. This conference is one among the series of mock conferences under the name of *Advance Theory and Application of Control Systems*, which is organised each year at various places in Japan by laboratories which are historically of the same family. ATACS-96 was participated by 14 research groups or laboratories from six different universities, with altogether 102 participants, 48 of whom gave a presentation. ATACS-97 has 42 paper presentations and 130 participants from 16 research groups of altogether ten universities. ATACS-98 was participated by 114 people from 13 research

groups of five universities, among whom 38 people were speakers. These are the three ATACS conferences I attended, namely the one in 1996 at Izu Peninsula in Shizuoka Prefecture and the one in 1998 at Lake Kawakuchi in Yamanashi Prefecture both of which are in the excellent setting of the Fuji-Hakone-Izu National Park, and in 1997 at a youth centre bordering on the lovable Yoyogi Park in the bustling Shibuya District of Tokyo.

To summarise, the RLS technique with random searching employed in § ℵ is developped further to accommodate back-stepping which helps produce a better result as shown in § ℵ. The latter paper has been presented at an international conference in Singapore (Tiyapan, 1997) and then as a seminar at TIT on 3^{rd} January 1998 . The former is the second joint conference of the International Simulation Societies called World Congress on Systems Simulation. It was organised by IEEE Singapore Section, the Society for Computer Simulation Europe, and the Society for Computer Simulation International. I gave a presentation on the last session, Session 21 on Potpourri, of the last day of the conference. As I did not buy the conference proceedings because it did not contain my paper except as inserted addendum pages copies of which I already had, I have no idea what the papers of the other presenters look like. The paper has been reviewed by two reviewers both of whom gave it a good for technical quality and an average for importance. For both the originality and the relevance to WCSS97, one gives a good while the other an average. The paper has been unanimously marginally accepted with the readability and presentation scoring somewhere between below average and average. One reviewer commented, 'The abstract is poor and too short. The set of references is also very poor. There is no conclusion to the paper! Previous remarks affect the quality of the paper, but it still remains in [sic] interesting study. The algorithm of the last section should be structed [sic]. Considering what I see of the paper I suggest that the author use a classical chart to present his algorithm. More precision (description) must be added to the functions an [sic] variables in equations. Some are described some are not. Before the algorithm section the author should give more precision concerning the random search and he could add references. Though my english is not very good, I think that the text should be improved (have it re-read by a native or eglish teacher if possible, or at least by another colleague.[sic]' The other gave a more succinct comment, 'The paper seems to be in-complete. The problem under consideration should be defined more clearly. Advantages of the proposed algorithm should be discussed in more details. Practical problems that could be solved by the proposed algorithm should be discussed. Conclusions should be added.' None of these suggestions I had followed because Dr. Helen D. Karatza the Chair for paper review wrote saying that IEEE Singapore would inform me about the final decision for my paper but I heard nothing from the latter although they told me later that they had sent a letter to me at the Furuta Laboratory, TIT. Interestingly enough I find a note in Japanese received from Ms. Shinata dated 7^{th} March just now among the document in I keep. She was then the secretary of Professor Furuta. The note says that he could not help with the cost of the conference since he has no idea what kind of paper I had written. Only now that it occurs to me that a few month earlier on I did actually present a similar paper which contains two-third of the contents in a conference which was organised by himself (*cf* Tiyapan, 1996, or § ℵ)! That aside, I have thus far come to the conclusion that the purpose of the organiser of a conference is in order to make money.

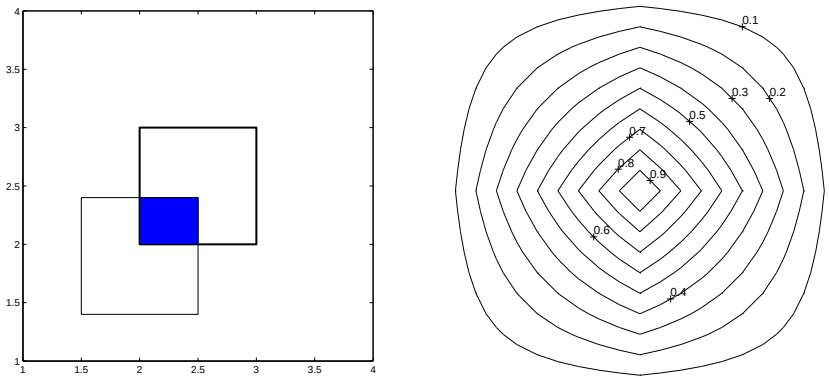
The following summarises my presentation at the WCSS97 conference. The original ideas of the research *RLS algorithm for object-location problems* come from an example in a design-exercise for an M.Sc. course in Control and Information Technology which I did with Dr. M. B. Zarrop at the Control System Centre, UMIST, during 1994–1995.

The same paper I also presented in a seminar of the Furuta Laboratory on 3^{rd} January 1998 . The manner of the presentation is recaptured in the following. The performance function of an object location problem can be represented by the area of the intersection between the object and and a randomly placed mask as shown in Figure 6.3 and 6.4. The former shows the hill positioned relative to the centre of gravity of the object, while the latter shows the contour of the hill resulted from a square object.



Both the object and the mask are within the domain D. The mask is placed in such a way as to maximise the area of its intersection with the object. The object and the mask may have any shape, and their shapes can differ.

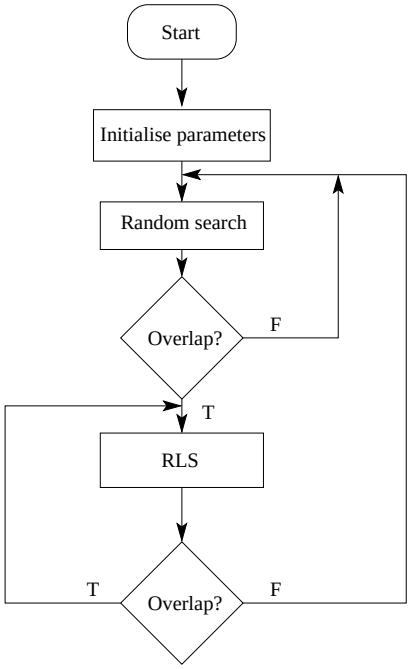
Figure 6.3 The hill of an object location problem.



(a) (b)

Figure 6.4 (a) *Area of the intersection between square object and a square mask,* (b) *contour of the hill thus obtained.*

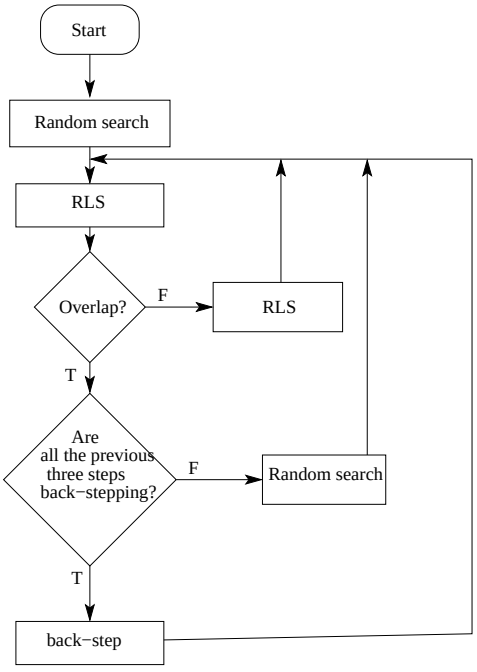
The procedure used in the paper I presented at the ATACS96 conference is shown in Figure 6.5.



This procedure does a random searching whenever the position of a hill is lost. Once the hill is found it switches over to a recursive least square subroutine.

Figure 6.5 RLS procedure with random search.

Then Figure 6.6 shows the improved algorithm in the paper presented today.



This is the same as the procedure in Figure 6.6 except that whenever losing sight of the hill it recursively steps back a step at a time for the maximum of three times or until the hill is found again. If the hill is still not found then it switches to do random searching as the other procedure does.

Figure 6.6 RLS procedure with random search and back-stepping.

7
Addenda

§ 7. Addenda

§ 7.1 Cyberspace

In 1983 Apple computers had been installed in number at the Ashburton College, New Zealand, where I was doing my Sixth Form as an American Field Service exchange student, and Pascal was taught. In 1993 I worked as a Security Officer for the TelecomAsia Company in Thailand. My job was to look after data on the three mainframe computers in use. It was there that I learnt terms like *cyberspace* and *hackers*. I became fascinated by this idea, that there is a virtual space tessellated into partitions which house a population whose identity is related to real people, because to me it is a whole new dimension added to the existing ones. Starting doing my master degree in Control and Information Technology at UMIST in 1994, I came in closer touch with the cyberspace community through the use of Unix is the first thing taught in that course. I voluntarily wrote two articles on the subject, the first one is about cyberspace (Tiyapan, 1994, § ℵ) and the second one about its security (Tiyapan, 1996, § ℵ).

§ 7.2 Sociology

The nature of discoveries and progresses in science is according to Bacon (Francis Bacon, 1620) *a birth of Time rather than a birth of Wit.* This is the same idea of percolation and the description he gave is the very picture of the theory. According to him major scientific progresses come in revolutions which are sparsely distributed in both time and regions. There have only been three periods of major progress out of the *five and twenty centuries over which the memory and learning of men extends*, namely the Greeks, the Romans and the nations of Western Europe. These are narrow limits of time, the periods in between of which are unfavourable to development. A discovery or an invention, then, comes as a chance accident in the scale of an individual, and as a certainty when looking from a distance.

When the time is right and all the hidden momentum built up, theories will come on by itself as a rule. This does not negate the excellence of an individual, but in a society where there are enough multitude of individuals the show will always go on, with or without a particular genius. This idea can very well explain cases of multiple discoveries. According to Kekulé in his *Benzolfest* speech in 1890, when he ascribed his conception of the cyclic nature of Benzene in dreams, *certain ideas at certain times are in the air* and *if one man does not enunciate them, other will do so soon afterwards.*

To see the relationship of this with percolation it is possible to look at two different things in turn, first at the discoveries and then at the discoverers. With a unit being that of a *discovery* the connection to percolation is that big discoveries come as connections of other smaller and less obvious ones. A theory often has more than one perspective, and which one of them comes to the fore first depends much on which combination happens to percolate through first. The discoveries of Schrödinger and Heisenberg in Quantum Physics can bear witness to this both in the *combination* and the *multiple discoveries* parts of this argument.

Let us turn our attention now to the scientist and look at the one who does the discovering instead. The theory of percolation tells us that at the point of discovery he is by no mean the sole integral ingredient. If he does not do it, then someone else will certainly do. In order to see this, I did four simulations for the cell, bond, vertice, and edge percolations on a two-dimensional Voronoi network and then another four with the same respective blockage of each case but considering the inverse phase instead. The number of units considered for the four cases are $n_c = 200$, $n_b = 416$, $n_v = 298$, and $n_e = 426$. With the order of simulations as described above, at just one step before percolation occurs there are respectively 10.6, 13.4, 11.5, 1.1, 11.24, 10.0, 19.4, and 7.9 percents among the remaining units which will readily trigger the onset of percolation. In other words, these are atoms which are able to link up existing clusters and form a percolating cluster.

The formation of mobs is an interesting phenomon comparable with phase change in physice. What happens is that an agglomerate of individuals becomes one and a single creature, the underlying mechanism of which still baffles any effort towards understanding it. Likely enough it has got something to do with psychology and the mind. But to me at least, the phenomenon is percolative. Having gained some acceptance from my previous writings (Tiyapan, 1995–1996, §'s ℵ, ℵ and ℵ) I gave my new work which briefly discusses the mechanism underlying the formation of mobs to an editor of the Sakkayaphab journal whom I know. At that time a political turmoil unequal in its degree and extent has been going on for five years. Whether by fate or by design there has been a successful but tragic use of mob in Bangkok. The word *mob* has joined the list of those synonym to

distrust, namely *communist* or, in western community now for that matter, *Islam* and *evil*. Whether because of this or something else, the article (Tiyapan, 1996, § ℵ) simply and mysteriously got lost; no one would admit having seen it, and the translation of another subsequent article of mine (1996, § ℵ) has not been without a noticeable negligence. Thus to me distrust is also percolative. The list of things one finds over-distrusted without reasonable explanation goes on indefinitely, homosexuality, communism, *etc.* The same seems to be the case with bad habits. My father used to teach me using the following poem,

> Bad habit gathers by unseen degree
> like brook makes river, river runs to sea.

Looking back, it could have been the title of that article, *on pragmatists and idealists*, which has somehow convinced the editors into believing that it was political which to me is nonsensical. I only meant literary, even if at times philosophical. I include it here because it contains a curve showing a critical emotional transition.

The formation of the United States, the European Union, or the Commonwealth comes from the trust which acts to join countries together like glue boxes in TEX. Like all binding forces, trust is mutual and spreads in the same way as a growing cluster does. The cluster grows bigger as one or more members are added, and it becomes stronger as the level of the mutual trust increases. In a similar way, distrust is also mutual and also spreads. If I distrust you and you distrust me, I will make sure that I remain as far away from you as possible while you will certainly avoid me by all means in return.

Only these two are possible, so there are only two phases to consider, that of trust and distrust. The relationship where one trusts while the other distrusts would not be stable, since the former will soon learn to join the latter. Trust forms clusters of one phase, distrust another. The size of these clusters vary in a way similar to those in percolation of geometrical networks. The strength of the glue is analogous to the probability either of becoming or remaining a member of a cluster.

§ 7.3 Control systems

The following are relevant subjects in control systems. I have one first degree and one master degree in the subject. From 1995 to 1999 I did a doctorate course in Tokyo but decided to quit after three years. Research in this area is still one of the topics that I would like to do in the future. One year before leaving Japan my research results must have look quite well because on 22^{nd} April 1998 my supervisor Professor Katsuhisa Furuta wrote me an email which says, 'Your reports are very nice! Excellent results. I am verymuch [sic] impressed. We can talk the results.', signed 'Furuta'. And again on 2^{nd} June 1998 another email, 'Dear mr.kit [sic] please come and explain your synchronous motor. Where shall we introduce the controller?', signed 'Katsuhisa Furuta'.

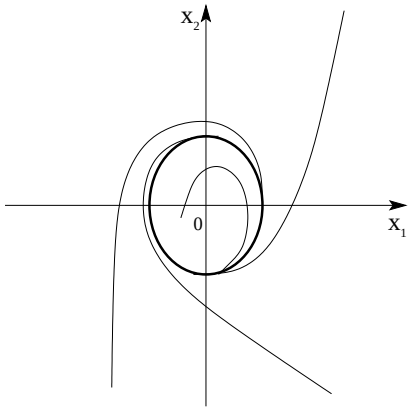
Although it is true that control systems is always used in military and misiles, one must not forget that it is also important for travelling in space and is likely to be of a great help to those space settlers searching for habitable planets in the future. Control system is used much in electrical, mechanical, and chemical engineering, with the typical time constant increasing in that order.

This section is in a way a brief recapture of what I did during my PhD study at TIT, Japan. I collected my works in a form of Technical Reports and gave a copy each to Professor Furuta who was my supervisor then. Sometimes when there was a spare copy left I would give it to another senior staff who worked in the Minami 5 building where the Furuta lab used to be. The computer files of these reports is no longer available even to myself. A great number of figures from simulation results in these reports are not reproducible here without the Simulink facility on Matlab. The first of such report was dated 16^{th} July 1998 but the works it contains started around the beginning of April of the same year. I studied systems listed in a book by Khalil (1996). One of these systems (Exercise 1.17 (4)) is

$$\dot{x}_1 = x_1 + x_2 - x_1\left(|x_1| + |x_2|\right), \qquad (\mathbf{2})_{\text{vii}}$$

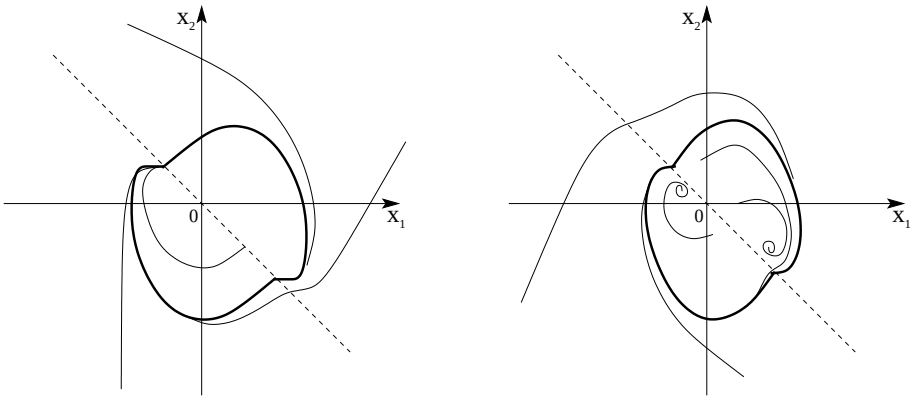
$$\dot{x}_2 = -2x_1 + x_2 - x_2\left(|x_1| + |x_2|\right) + u, \qquad (\mathbf{3})_{\text{vii}}$$

where u is the input to the system and x_1 and x_2 the state variables. Because the simultaneous equations $x_1 + x_2 - x_1\left(|x_1| + |x_2|\right) = 0$ and $-2x_1 + x_2 - x_2\left(|x_1| + |x_2|\right) = 0$ are ill-formed, there can be no equilibrium points.



The limit cycle in Figure 7.1 is elliptical with the major axis along the x_2-axis. When the control input is $u = \text{sgn}(x_1 + x_2)$ the phase plane looks like Figure 7.2 (a) and when $u = -(x_1 + x_2) + \text{sgn}(x_1 + x_2)$ Figure 7.2 (b).

Figure 7.1 Phase plane of the system described by Equations 7.2 and 7.3 when there is no input.



(a) (b)

When there is no control Figure 7.2 shows that the equilibrium point at the origin is an unstable focus and there is a limit cycle circling around the origin. This limit cycle is slightly larger in the x_2 direction than in the x_1 direction. Every trajectory starting from an initial point other than the origin goes to and then stays on this limit cycle. With the control input $u = \text{sgn}(x_1 + x_2)$ there is a discontinuity on the surface of the hyperplane $s = x_1 + x_2 = 0$. All trajectories still converge to the limit cycle although the latter is distorted where it intersects the hyperplane. There is no node and every point on the hyperplane and inside the limit cycle is an unstable node. All these also hold when the control input is $u = -(x_1 + x_2) + \text{sgn}(x_1 + x_2)$ and there are two additional nodes as shown in Figure 7.2 (b).

During September and October 1996 I gave a series of seminars on a topic related to polytopes of polynomials. The topic I chose was recent (Pujara, 1996) and rather difficult for me but for me it was a success because the methods introduced was discussed weekly for at least a month in a series of the subsequent seminars, and they resulted in the idea being successfully applied, in the context of Pulse Width Modulation, by one of the students who attended in his Ph.D. work. The following is a recapture of the original seminar I gave. The discussions, which was the most interesting part in these seminars, are lost. A pseudoboundary is defined to be the set of all polynomials in the polytope each of which has at least one zero on the imaginary axis. A section of the pseudoboundary corresponding to ω_0 is a polytope whose vertices lie in the exposed 2-d faces of the given polytope. Pujara (1996) in his study of the stability boundary problem gave an algorithm to generate all the vertex polynomials of any section of the pseudoboundary of a polytope. A polytope is stable if and only if its exposed edges are stable. Interval polynomials are a hyperrectangle in coefficient space with edges parallel to the coordinate axis. Kharitonov's results is that if a polynomial family consists of interval polynomials, then the stability of just four extreme polynomials is both necessary and sufficient for the stability of the entire rectangle. Consider the polynomial $f(s,q) = s^n + a_1(q)s^{n-1} + a_2(q)s^{n-2} + \ldots + a_{n-1}(q)s + a_n(q)$ which produces an r-dimensional polytope P in R^{n+1}. Here $a_i(q)$, $i = 1$ to n, are real affine coefficients and $q_i^- \leq q_i \leq q_i^+$ for every q_i, $1 \leq i \leq r$. For fixed q, this polynomial is a point in R^{n+1} whose coordinates are the coefficients of the polynomials. The pseudoboundary, β, is the set of polynomials in the polytope P each of which has at least one zero on the imaginary axis. The section, β_0, of β at ω_0 consists of those polynomials each of which has a zero at $j\omega_0$. If $Z \in C^{n+1}$ is a set of all zeros of this polynomial, then $\omega_0 \in W$ if and only if $\exists g(j\omega_0) \in P$ such that $g(j\omega_0) = 0$. Every $\beta(j\omega_0)$ is a polytope which has its vertices on the exposed 2-d faces of P, which in turn can be explicitely determined.

Theorem. *(Pujara, 1996) The section of the pseudoboundary β of a polytope P at any frequency ω_0 is a polytope. The vertices of this pseudoboundary lie in the exposed 2-d faces of the polytope P.*

The state equations of a dc motor are

$$J\frac{d\omega}{dt} = ki \qquad (4)_{\text{vii}}$$

$$L\frac{di}{dt} = -k\omega - Ri + u \qquad (5)_{\text{vii}},$$

where i is the armagure current, u the voltage, R the resistance, L the inductance, J the moment of inertia, and ω the angular speed. The constant excitation flux ϕ results in the torque ki and the back e.m.f. $k\omega$.

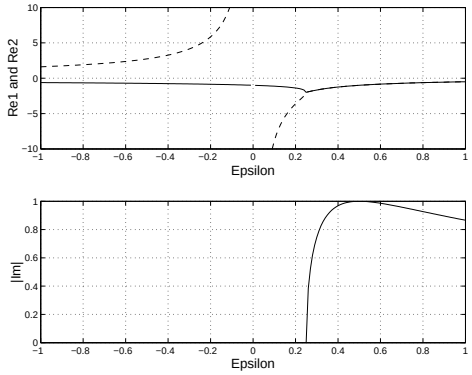
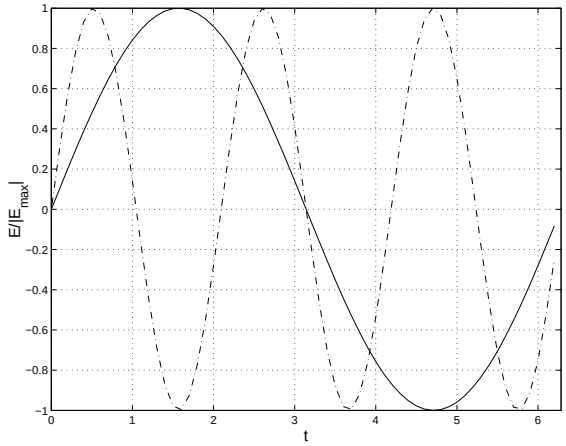


Figure 7.3 The eigenvalues of the dc motor as a function of ε.

My technical report number 6 (1998) I studied the control of synchronous machines. In the introduction I wrote to Professor Furuta to say that he may use any of the material without acknowledging me. I see it fits to describe briefly here some of the results because I initiated and carried out the work, as is normally the case with most of my researches done while in Japan. The electrical frequency of such machines synchronises with the mechanical speed. However it is more convenient to express angles in electrical unit, *i.e.* electrical angle, rather than in mechanical unit, *i.e.* mechanical angle, for the reason that a synchronous machine can have more than two poles. Whichever is the case, one may consider only a single pair of poles and then consider the electrical, mechanical, and magnetic conditions, which are associated with all the other pole pairs as repetitions of those for the pair being considered (*cf* Fitzgerald *et al*, 1971).

The electrical and the mechanical angles of a generator are related by the relation $\theta_e = \frac{p}{2}\theta_m$, where θ_e is the electrical angle of the output, θ_m is the angle of the rotating shaft, and p is the number of poles of the machine. The frequency of the voltage is $f = \frac{p}{2}\frac{n}{60}$ Hz, where n is the mechanical speed in rpm and $\frac{n}{60}$ the speed in revolution per second. The radian frequency ω of the voltage is $\omega_e = \frac{p}{2}\omega_m$, where ω_m is the mechanical speed in radian per second.



The mechanical and the electrical rotation of a synchronous machine are juxtaposed in Figure 7.4. In power system engineering one wants to control teh power and the frequency. Examples of controllers are speed govenors, actuators, regulators, and signal transducers. Speed govenors can be described by $\Delta f \cdot K = T_L$, where Δf is the change in frequency, K the gain, and T_L the load torque. The other controllers mentioned are high-gain power-amplifiers which convert things like position, oil pressure, and electricity into valve positions.

Figure 7.4 Dynamics of the mechanical and the electrical angles of a synchronous machine with three pair of poles. The solid line is the mechanical, while the dash-dot line the electrical rotation.

The change in load in the power-frequency control of a synchronous generator can either be a *load shedding* where the load decreases causing the frequency to increase, a *generator shedding* where the load increases causing the frequency to decrease, or a *short circuit* where the generator is suddenly disconnected and as a result the frequency increases to the maximum value. The effects

caused by the load change are related to the equation $J\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = T_{PM} - T_E$, where J is the moment of inertia of the rotating mass, θ the angular position of the rotating mass, T_{PM} the prime-mover torque, and T_E the electrical torque.

In 1998 I presented a paper (§ ℵ) at the ATACS98 conference in Japan. The objectives of that study are to study the effects of a sign function input, to study the effects of ε in a singularly perturbed system, and to study a variable structure singularly perturbed system in other words those which have a sign function in the control input. *Singular perturbation* of any system implies that it has ε_i terms multiplying the term $\dot{\mathbf{x}}$ and that these terms ε_i approach zero. One can write down a given transfer function $T(s)$ in the state space forms $\dot{\mathbf{x}} = \mathbf{E}^{-1}\mathbf{A}\mathbf{x} + \mathbf{E}^{-1}\mathbf{B}\mathbf{u}$ and $y = \mathbf{C}\mathbf{x} + D$. For example, $T(s) = \frac{2(s+5)(s^2+4s+29)}{(s+10)(s+1)(s^2+8s+20)}$ can be represented by the above state equations with

$$\mathbf{A} = \begin{bmatrix} -8 & -5 & 1 & 1.25 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & -11 & -2.5 \\ 0 & 0 & 4 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 2.82843 \\ 0 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{C} = [-2.83 \quad 1.6 \quad 0.71 \quad 0.88], \quad \text{and} \quad D = 0$$

Then if we rewrite $\mathbf{E}$ as

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \varepsilon \end{bmatrix},$$

we can study the perturbation effect of the singularly in ε as it decreases from $\varepsilon < 1$ towards $\varepsilon \ll 1$, namely

$$\begin{aligned}
\varepsilon = 0.5, T(s) &= \frac{2(s+10)(s^2+4s+29)}{(s^2+8s+20)(s+2.2984)(s+8.7015)} \\
\varepsilon = 0.1, T(s) &= \frac{2s^3+108s^2+458s+2900}{(s^2+8s+20)(s^2+11s+100)} \\
\varepsilon = 0.01, T(s) &= \frac{2s^3+1008s^2+4058s+29000}{(s^2+8s+20)(s^2+11s+1000)} \\
\varepsilon = 0.001, T(s) &= \frac{2s^3+10008s^2+40058s+(2.9\times 10^5 2.9)}{s^4+19s^3+10108s^2+80220s+(2\times 10^5)} \\
\varepsilon = 0.0001, T(s) &= \frac{2s^3+100008s^2+400058s+(2.9\times 10^6)}{s^4+19s^3+(1.0011\times 10^5)s^2+(8.0022\times 10^5)s+(2\times 10^6)}.
\end{aligned}$$

To summarise the root locus behaviour when ε decreases, two complex zeros are fixed, that is $s^2+4s+29$. Two complex poles described by $s^2+8s+20$ remain the same throughout. The zero on the real axis moves left. The two poles on the real axis moves toward each other. At the point when the moving zero passes the pole on its left hand side the root locus changes its characteristics.

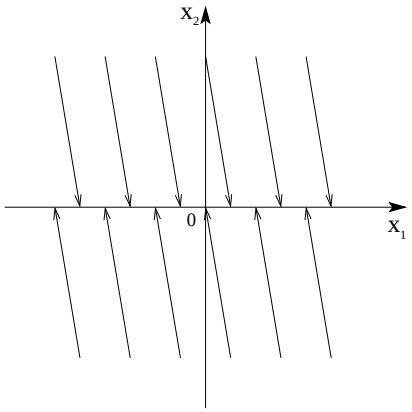
Even as early as November 1998 I have decided that Control Systems will not be the only area that I will do researches in, as can be seen in the title of the technical report number two (1998) which has the words *system and control* put in brackets. Here among other things I considered the system described by the differential equation

$$\varepsilon\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{\mathrm{d}x}{\mathrm{d}t} = u \qquad (\mathbf{6})_{\text{vii}}$$

. This can be rewritten as the state equations $\dot{x}_1 = x_2$ and $\varepsilon\dot{x}_2 = -x_2 + u$ which represent a model in the *standard form* since the second equation has a real isolated root when $\varepsilon = 0$. By letting the right hand side of the state equation be zero, every point on the x_1 axis is an equilibrium point. A model described by the equations $\dot{x} = f(t,x,z,\varepsilon)$, $x \in R^n$ and $\varepsilon\dot{z} = g(t,x,z,\varepsilon)$, $z \in R^n$ is said to be in the standard form if and only if $0 = g(t,x,z,0)$ has k isolated real roots $z = h_i(t,x)$, $i = 1,2,\ldots,k$ (Khalil, 1996); then these two equations reduces to a *quasi-steady state* or *slow model* $\dot{x} = f(t,x,h(t,x),0)$.

For the present system the eigenvalues obtained from the Jacobian matrix $A = \left.\frac{\partial f}{\partial x}\right|_{x=(x_1,0)} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{1}{\varepsilon} \end{bmatrix}$ are 0 and $-\frac{1}{\varepsilon}$. Therefore it has an equilibrium subspace and the qualitative behaviour of the trajectories depends on the values of ε, that is when $\varepsilon > 0$ all trajectories converge to the equilibrium subspace, when $\varepsilon = 0$ the system degenerates and is reduced to a first order system, and

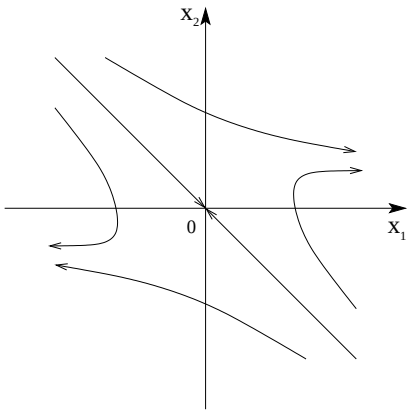
when $\varepsilon < 0$ then all trajectories diverge from the equilibrium subspace. In other words the system is stable when $\varepsilon > 0$ and unstable when $\varepsilon < 0$. By plotting the state planes and the time response of the state variables of this system when $u = 0$, one can see that the system is stable if $\varepsilon > 0$, and unstable if $\varepsilon < 0$. When $|\varepsilon|$ becomes smaller the response of the system becomes more rapid as $\dot{x}_2 = \frac{1}{\varepsilon} x_2$ rapidly converges to its root $x_2 = 0$ when ε approaches zero. The trajectory is a straight line which becomes more parallel to the x_2-axis the closer ε gets to zero.



As ε approaches zero and $u = 0$ the system becomes $x_2 = 0$ or $\dot{x}_1 = 0$ which has the solution $x_1(t) = x_1(0)$ and $x_2(t) = 0$. The singular point $\varepsilon = 0$ is the point of discontinuity since the solution is stable when $\varepsilon \to 0^+$ and unstable when $\varepsilon \to 0^-$.

Figure 7.5 State plane of the system $\varepsilon \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{\mathrm{d}x}{\mathrm{d}t} = u$ with $0 < \varepsilon < 1$.

Next study the effect of a feed back with the sign function, $u = K\,\mathrm{sgn}(Cx_1 + x_2)$, to the system in Equation 7.6, where C and K are real constants. When $C = K = 1$, simulation results showed that the response of the controlled system changes more rapidly as $|\varepsilon|$ becomes smaller. In other words, the response becomes faster as ε approaches zero. This increase in speed of the response as ε decreases in magnitude applies for the unstable case where $\varepsilon < 0$ as well as when $\varepsilon > 0$.



The boundary layer or the transient period of the system remains the same either with or without the input, that is t of approximately $[0, 0.05]$ and $[0, 0.4]$ for $\varepsilon = 0.01$ and 0.1 respectively. For $\varepsilon \geq 1$ this boundary layer extends beyond one second. With the feed back input, the state plane trajectory reaches the line $x_2 = 0$ during the transient period and then gently slides along it.

Figure 7.6 State plane of $\varepsilon \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{sgn}(x_1 + x_2)$ when $\varepsilon \geq 1$.

The state equations with the input become $\dot{x}_1 = x_2$ and $\varepsilon \dot{x}_2 = -x_2 + \mathrm{sgn}(x_1 + x_2)$. When $\varepsilon = 0$ the latter equation above becomes $0 = -x_2 + \mathrm{sgn}(x_1 + x_2)$ which gives the root $\bar{x}_2 = \mathrm{sgn}(\bar{x}_1 + \bar{x}_2)$ that can not be written in the form $\bar{x}_2 = h(t, \bar{x}_1)$. This means that the equation has no isolated

root and consequently one can not isolate the fast mode from the slow mode by introducing a new variable $y = x_2 - h(t, x_1)$. But for the purpose of finding the boundary layer model, let $h(t, x_1, x_2) = \text{sgn}(x_1 + x_2)$. Then $y = x_2 - h(t, x_1, x_2) = x_2 - \text{sgn}(x_1 + x_2)$ and as a consequence $x_2 = y + \text{sgn}(x_1 + x_2)$. Introduce a new time variable τ obtained from $\varepsilon \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}\tau}$ or $\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{1}{\varepsilon}$ when $\tau_0 = 0$, that is $\tau = \tau_0 + \int_{t_0}^{t} t\mathrm{d}t = \frac{t - t_0}{\varepsilon}$. This new time variable τ is generally known as the *stretched* time variable. Then the equation $\varepsilon \left[\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\text{sgn}(x_1 + x_2)\right] = -y$ becomes $\frac{\mathrm{d}y}{\mathrm{d}\tau} + \frac{\mathrm{d}}{\mathrm{d}\tau}\text{sgn}(x_1 + x_2) = -y$. Substituting $x_2 = y + \text{sgn}(x_1 + x_2)$ gives $\frac{\mathrm{d}y}{\mathrm{d}\tau} = -y - \frac{\mathrm{d}}{\mathrm{d}\tau}\text{sgn}(x_1 + y + \text{sgn}(x_1 + x_2))$, and therefore the boundary layer model is $\frac{\mathrm{d}y}{\mathrm{d}\tau} = -y - \frac{\mathrm{d}}{\mathrm{d}\tau}\text{sgn}(x_1 + x_2 + 1)$ when $x_1 + x_2 > 0$, is $\frac{\mathrm{d}y}{\mathrm{d}\tau} = -y - \frac{\mathrm{d}}{\mathrm{d}\tau}\text{sgn}(x_1 + x_2)$ when $x_1 + x_2 = 0$, or is $\frac{\mathrm{d}y}{\mathrm{d}\tau} = -y - \frac{\mathrm{d}}{\mathrm{d}\tau}\text{sgn}(x_1 + x_2 - 1)$ when $x_1 + x_2 < 0$. When $\bar{x}_1 + \bar{x}_2 > 0$ then $\bar{x}_2 = +1$ and the reduced problem becomes $\dot{x}_1 = +1$. Likewise when $\bar{x}_1 + \bar{x}_2 = 0$ then $\bar{x}_2 = 0$ and $\dot{x}_1 = 0$, and when $\bar{x}_1 + \bar{x}_2 < 0$ then $\bar{x}_2 = -1$ and $\dot{x}_1 = -1$.

The discontinuous nature of the signum function makes the analysis of the variable structure control system difficult. This is due to the fact that one can not analytically find $\frac{\partial \text{sgn}\, x(t)}{\partial t}$, $\frac{\partial \text{sgn}\, x(t)}{\partial x_i}$, where $i = 1, 2, \ldots, n$ and $x(t)$ is an n-tuple vector. One way to overcome or go around this problem is to use a numerical approximation for the sign function where necessary.

Let the input be $u = -(x_1 + x_2) + \text{sgn}(x_1 + x_2)$, then simulations show that the system has two timescales, which is the characteristic of a singularly perturbed system. Having a fast response within the boundary layer and a slow response elsewhere, the boundary layer becomes narrower and the response faster as ε approaches zero from above ($\varepsilon \to 0^+$). Also, numerical studies shows that this boundary layer, in seconds, is approximately $[0, 0.05]$, $[0, 0.23]$, and $[0, 0.54]$ respectively when ε is 0.01, 0.1, and 1. With this input, the equations of the system become $\dot{x}_1 = x$ and $\varepsilon \dot{x}_2 = -x_1 - 2x_2 + \text{sgn}(x_1 + x_2)$. The last equation becomes $= 0 - x_1 - 2x_2 + \text{sgn}(x_1 + x_2)$ when $\varepsilon = 0$, in other words $x_2 = -\frac{1}{2}x_1 + \frac{1}{2}\text{sgn}(x_1 + x_2)$. To find the boundary layer, change the variable x_2 to $y = x_2 - x_2|_{\varepsilon=0} = x_2 + \frac{1}{2}x_1 - \frac{1}{2}\text{sgn}(x_1 + x_2)$. Therefore $x_2 = y - \frac{1}{2}x_1 + \frac{1}{2}\text{sgn}(x_1 + x_2) = y - \frac{1}{2}x_1 + \frac{1}{2}\text{sgn}\left(x_1 + y - \frac{1}{2}x_1 + \frac{1}{2}\text{sgn}(x_1 + x_2)\right)$. Which is essentially that x_2 equals $y - \frac{1}{2}x_1 + \frac{1}{2}\text{sgn}(y + \frac{1}{2}x_1 + \frac{1}{2})$ when $x_1 + x_2 > 0$, $y - \frac{1}{2}x_1 + \frac{1}{2}\text{sgn}(y + \frac{1}{2}x_1)$ when $x_1 + x_2 = 0$, and $y - \frac{1}{2}x_1 + \frac{1}{2}\text{sgn}(y + \frac{1}{2}x_1 - \frac{1}{2})$ when $x_1 + x_2 < 0$. Substitute this into the equation for $\varepsilon \dot{x}_2$ above to get the boundary layer model $\varepsilon \left[\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{1}{2}\frac{\mathrm{d}x_1}{\mathrm{d}t} + \frac{1}{2}\text{sgn}(x_1 + x_2)\right] = -2y$ or $\varepsilon \left[\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{y}{2} + \frac{1}{4}x_1 - \frac{1}{4}\text{sgn}(x_1 + x_2) + \frac{1}{2}\text{sgn}(x_1 + x_2)\right] = -2y$. Use the stretched time variable τ introduced above. Then $\frac{\mathrm{d}y}{\mathrm{d}\tau} = -\frac{3}{2}y - \frac{1}{4}x_1 - \frac{1}{4}\text{sgn}(x_1 + x_2)$, which is the same as saying that $\frac{\mathrm{d}y}{\mathrm{d}\tau}$ equals $-\frac{3}{2}y - \frac{1}{4}x_1 - \frac{1}{4}$ when $x_1 + x_2 > 0$, $-\frac{3}{2}y - \frac{1}{4}x_1$ when $x_1 + x_2 = 0$, and $-\frac{3}{2}y - \frac{1}{4}x_1 + \frac{1}{4}$ when $x_1 + x_2 < 0$.

Now for a variable structure control, let the hyperplane be described by $s = cx_1 + x_2$ whose derivative is $\dot{s} = c\dot{x}_1 + xd_2$. Let the input be $u = -K\,\text{sgn}\, s = -K\,\text{sgn}(cx_1 + cx_2)$ for $K \in R^+$. Choose a Lyapunov function as $V = \frac{1}{2}s^2$. The state equations become $\dot{x}_1 = x_2$ and $\dot{x}_2 = -\frac{1}{\varepsilon}x_2 - \frac{K}{\varepsilon}(cx_1 + x_2)$. The derivative of the Lyapunov function is then $\dot{V} = s\dot{s} = (cx_1 + x_2)\left(cx_2 - \frac{1}{\varepsilon}x_2 - \frac{K}{\varepsilon}(cx_1 + x_2)\right) = c^2x_1x_2 - \frac{cx_1x_2}{\varepsilon} + cx_2^2 - \frac{x_2^2}{\varepsilon} - \frac{cKx_1}{\varepsilon}\text{sgn}(cx_1 + x_2) - \frac{Kx_2}{\varepsilon}(cx_1 + x_2)$. In order to observe the effect of ε on the stability of the controlled system, let $c = 2.7$ and $K = 3.3$. From plots of $\dot{V}$ against x_1 and x_2 one may see that whether $\dot{V}$ be positive or negative depends upon the value ε takes. For example, there exists x which gives $\dot{V} > 0$ when ε is 1.7 and 0.08, while if $\varepsilon = 0.3$ then $\dot{V} < 0$ for all possible values of x.

The original system (Equation 7.6) has the eigenvalues at $\lambda_1 = 0$ and $\lambda_2 = -12.5$, with the corresponding eigenvectors at $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -0.0797 \\ 0.9968 \end{bmatrix}$ respectively. The system is linear since A is a constant matrix and does not depend upon x. Also A is not a stability matrix since the condition $\text{Re}\,\lambda_i < 0$ is not satisfied. The equilibrium point at the origin is not unique as it has already been pointed out earlier that every point on the x_1-axis is an equilibrium point. The x_1-axis is a nontrivial null space of the matrix A. The change of variables $z = M^{-1}x$, where $M = [v_1 \quad v_2] = \begin{bmatrix} 1 & -0.0797 \\ 0 & 0.9968 \end{bmatrix}$, leads to $M\dot{z} = AMz = \begin{bmatrix} 0 & 0 \\ 0 & -12.5 \end{bmatrix}$, and therefore $A = \begin{bmatrix} 0 & 0 \\ 0 & -12.5 \end{bmatrix}$. The solution to this system is $\dot{z}_1(t) = z_1(0)$ and $z_2(t) = z_2(0)e^{-12.5t} = z_2(0)e^{\lambda_2 t}$. That A is no stability matrix can be seen by trying to solve the Lyapunov equation $PA + A^TP = -Q$, where Q is a positive definite symmetric matrix. The Lyapunov equation for this system yields no solution.

From simulation results, when $\varepsilon < 0$ the controlled system is unstable, while for $\varepsilon > 0$ increasing K will widen the range of ε in which the controlled system is stable. That is, the larger the value of

the gain K of the control input, the more robust with respect to ε this control scheme.

Note also that one way to stabilise the origin of the system is to solve the equation $A^TPE + E^TPA-(E^TPB+S)R^{-1}(B^TPE+S^T)+Q = 0$, or equivalently $F^TPE+E^TPF-E^TPBR^{-1}B^TPE+Q-SR^{-1}S^T = 0$, where $F = A-BR^{-1}S^T$, Q and R are symmetric positive definite matrices, and the system is written in the form $E\dot{x} = Ax + Bu$. For example, letting $Q = R = E = I$ and $S = 0$ leads to $P = \begin{bmatrix} 12.62 & 1 \\ 1 & 0/1194 \end{bmatrix}$. The feedback input is then $u = -Gx$, where $G = R^{-1}(B^TPE + S^T)$. The gain matrix will be $G = [1 \quad 0.1194]$. Then the closed loop eigenvalues can be computed from $(A - BG)V = EV\Gamma$, where $V = [v_1 \quad v_2]$, $\Gamma = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, λ_i is the i^{th} eigenvalue, and v_i its corresponding eigenvector. In solving the Riccati equation the Frobenius norm of the relative residual matrix is 3.14×10^{-16}. Simulations on Simulink show that when $\varepsilon = 0$ the system is unstable. There exists a domain of attraction outside which the trajectory does not converge.

Programs

§ A. Programs

§ 7.4 Object location

```
% Object2dSqua.m
clear all; figure(1); clf; a=1; b=2; c=3;
a1=1; a2=1; c1=10; c2=10; b1=-2*a1*5; b2=-2*a2*7; P=eye(6); lambda=.98;
DithVar=.1; NoiseVar=.1; Limit=10; N=1000; x=2; y=3;
[Object,m] =ObjSqu(x,y);
subplot(2,2,1),Draw(Object,m,Limit,'Object','c');
xlabel('X'),ylabel('Y'); axis equal; axis([-10 10 -10 10]);
subplot(2,2,2),Layout(Object,Limit,m);
Real=RealLin(x,y,N); zeta=[a1,a2,b1,b2,c1,c2]'; t=0; s2=0; s3=0;
for Steps=1:N
  v =dither(DithVar); p =-b1/a1/2 +v; q =-b2/a2/2 +v;
  [Image,n] =ImgMtx(round(p),round(q));
  if Steps==10
    Move1(:,1) =Image(:,1); Move1(:,2) =Image(:,2);
  elseif Steps==100
    Move2(:,1) =Image(:,1); Move2(:,2) =Image(:,2);
  elseif Steps==200
    Move3(:,1) =Image(:,1); Move3(:,2) =Image(:,2);
  elseif Steps==300
    Move4(:,1) =Image(:,1); Move4(:,2) =Image(:,2);
  elseif Steps==400
    Move5(:,1) =Image(:,1); Move5(:,2) =Image(:,2);
  elseif Steps==500
    Move6(:,1) =Image(:,1); Move6(:,2) =Image(:,2);
  elseif Steps==600
    Move7(:,1) =Image(:,1); Move7(:,2) =Image(:,2);
  elseif Steps==700
    Move8(:,1) =Image(:,1); Move8(:,2) =Image(:,2);
  elseif Steps==800
    Move9(:,1) =Image(:,1); Move9(:,2) =Image(:,2);
  end;
  g =Compare(Object,Image,m,n); e =noise(NoiseVar); y =g+e;
  if y>1
    x =[p*p,q*q,p,q,1,1]'; t =t+1; Est(Steps,1) =t;
    Est(Steps,2) =-b1/a1/2; s2=Est(Steps,2);
    Est(Steps,3) =-b2/a2/2; s3=Est(Steps,3);
    P =1/lambda*P*(eye(6)-(x*x'*P)/(lambda+x'*P*x));
    zeta =zeta+P*x*(y-x'*zeta); a1 =zeta(1,:); a2 =zeta(2,:);
    b1 =zeta(3,:); b2 =zeta(4,:);
  else
    t =t+1; Est(Steps,1) =t; Est(Steps,2) =s2; Est(Steps,3) =s3;
    p =RandSrch(Limit); q =RandSrch(Limit); b1 =-2*a1*p; b2 =-2*a2*q;
  end;
end;
Title1 ='X'; Title2 ='Y'; Title3 ='Phase Plane'; Title4 ='Object and Image';
XLab ='Steps'; YLab ='Mag';
subplot(2,2,3),compare2([Est(:,1),Est(:,2)],[Real(:,1),Real(:,2)],Title1,XLab,YLab)
subplot(2,2,4),compare2([Est(:,1),Est(:,3)],[Real(:,1),Real(:,3)],Title2,XLab,YLab)
figure(2); clf;
subplot(3,3,1),Draw(Object,m,Limit,'Object','r'); axis equal; axis([-10 10 -10 10]);
subplot(3,3,2),Draw(Move1,n,Limit,'N = 10','y'); axis equal; axis([-10 10 -10 10]);
subplot(3,3,3),Draw(Move2,n,Limit,'N = 100','y'); axis equal; axis([-10 10 -10 10]);
subplot(3,3,4),Draw(Move3,n,Limit,'N = 200','y'); axis equal; axis([-10 10 -10 10]);
subplot(3,3,5),Draw(Move4,n,Limit,'N = 300','y'); axis equal; axis([-10 10 -10 10]);
subplot(3,3,6),Draw(Move5,n,Limit,'N = 400','y'); axis equal; axis([-10 10 -10 10]);
subplot(3,3,7),Draw(Move6,n,Limit,'N = 500','y'); axis equal; axis([-10 10 -10 10]);
subplot(3,3,8),Draw(Move7,n,Limit,'N = 600','y'); axis equal; axis([-10 10 -10 10]);
subplot(3,3,9),Draw(Move8,n,Limit,'N = 700','y'); axis equal; axis([-10 10 -10 10]);
 % Compare.m
function [y] =Compare(Object,Image,m,n)
y =0;
for i=1:m
  for j=1:n
    if (Object(i,1)==Image(j,1))&(Object(i,2)==Image(j,2))
      y =y +1;
    end;
  end;
end;
 % compare2.m
function [] =compare2(data1,data2,graph_title,x_label,y_label)
plot(data1(:,1),data1(:,2),'-',data2(:,1),data2(:,2),'-');
grid on; title(graph_title); xlabel(x_label); ylabel(y_label);
 % dither.m
function [dither]=dither(dith_variance)
dith_limit=sqrt(3*dith_variance); dither=dith_limit*2*(.5-rand(1));
 % Draw.m
```

```
77 function []=Draw(Pict,Dim,Limit,Title,Colour)
78 for i=1:Dim
79 fill([(Pict(i,1)-.5),(Pict(i,1)+.5),(Pict(i,1)+.5),(Pict(i,1)-.5)],...
80   [(Pict(i,2)-.5),(Pict(i,2)-.5),(Pict(i,2)+.5),(Pict(i,2)+.5)],Colour),hold on;
81 end;
82 axis([-Limit Limit -Limit Limit]); grid; title(Title);
83  % ImgMtx.m
84 function [yy,n]=ImgMtx(p,q)
85 i=1;
86 for x=(p-1):(p+1)
87   for y=(q-1):(q+1)
88     yy(i,1)=x; yy(i,2)=y;i=i+1;
89   end;
90 end;
91 n =size(yy,1);
92  % Layout.m
93 function []=Layout(Object,Limit,m)
94 for i=-Limit:Limit
95   x =i+Limit+1;
96   for j=-Limit:Limit
97     y =j+Limit+1; [Image,n] =ObjMtx(i,j); xx(x,1) =i; yy(y,1) =j;
98     z(x,y) =Compare(Object,Image,m,n);
99   end;
100 end;
101 mesh(xx,yy,z); title('Object Profile'); xlabel('x'); ylabel('y');
102 zlabel('Pixels overlapped'); grid;
103  % noise.m
104 function [e]=noise(noise_sig)
105 e=sqrt(noise_sig)*randn(1);
106  % ObjSqu.m
107 function [yy,n]=ObjSqu(p,q)
108 i=1;
109 for x=(p-1):(p+1)
110   for y=(q-1):(q+1)
111     yy(i,1) =x; yy(i,2) =y; i =i+1;
112   end;
113 end;
114 n =size(yy,1);
115  % RealLin.m
116 function [Real] =RealLin(x,y,N)
117 for i=1:N
118   Real(i,1) =i; Real(i,2) =x; Real(i,3) =y;
119 end;
120 % RandSrch.m
121 function [y]=RandSrch(limit)
122 y =limit*2*(.5-rand(1));
```

§ 7.5 Continuum percolation of n-gons

```
% regpoly.m
clear all; St=sum(100*clock); rand('state',St); Size=10; N=40; n=11; Angle=2*pi/n;
X=Size*rand(N,2); TwoPi=2*pi; Rad=TwoPi*rand(N,1);
R=sqrt(1/(n*sin(Angle/2)*cos(Angle/2)));
for i=1:N,
  for j=0:(n-1),
    Tmp=Rad(i,1)+j*Angle; V{i}{1,(j+1)}=[(X(i,1)+R*cos(Tmp)),(X(i,2)+R*sin(Tmp))];
  end
end
Tmp=V{1}{1,1}+(V{1}{1,2}-V{1}{1,1})/2; dx=Tmp(1,1)-X(1,1); dy=Tmp(1,2)-X(1,2);
r=sqrt(dx*dx + dy*dy); clf; hold on;
for i=1:N,
  x=[]; y=[];
  for j=1:n,
    x=[x;V{i}{1,j}(1,1)]; y=[y;V{i}{1,j}(1,2)];
  end
  x=[x;V{i}{1,1}(1,1)]; y=[y;V{i}{1,1}(1,2)]; plot(x,y);
end
plot([0,Size,Size,0,0],[0,0,Size,Size,0]); axis equal; axis off;
for i=1:N,
  Tmp=[];
  for j=1:n,
    Tmp=[Tmp;[j,V{i}{1,j}(1,2)]]; Tmp=sortrows(Tmp,2);
  end
end
T=delaunay(X(:,1),X(:,2)); NT=size(T,1); D=sparse(N,N); Ov=sparse(N,N);
Ov1=sparse(N,1); Pair=[]; Limbo=[]; Oclock=sparse(NT,NT);
for i=1:NT,
  Tmp=[T(i,:),T(i,1)];
  for j=1:3,
    c1=Tmp(1,j); c2=Tmp(1,(j+1)); dx=X(c2,1)-X(c1,1); dy=X(c2,2)-X(c1,2);
    TmpA=sqrt(dx*dx + dy*dy); D(c1,c2)=TmpA; D(c2,c1)=TmpA;
    Pair=[Pair;[c1,c2;c2,c1]];
    if(TmpA<=(2*r))
      Ov(c1,c2)=1; Ov(c2,c1)=1; Ov1(c1,1)=1; Ov1(c2,1)=1;
    elseif(TmpA<=(2*R))
      Limbo=[Limbo;[c1,c2;c2,c1]];
    end
  end
  TmpB=atan(abs(dy/dx));
  if(dx>=0)
    if(dy>=0)  % Quadrant 1
      Oclock(c1,c2)=TmpB; Oclock(c2,c1)=pi+TmpB;
    else  % Quadrant 4
      Oclock(c1,c2)=TwoPi-TmpB; Oclock(c2,c1)=pi-TmpB;
    end
  else
    if(dy>=0)  % Quadrant 2
      Oclock(c1,c2)=pi-TmpB; Oclock(c2,c1)=TwoPi-TmpB;
    else  % Quadrant 3
      Oclock(c1,c2)=pi+TmpB; Oclock(c2,c1)=TmpB;
    end
  end
end
Tmp=Angle/2; Star=[];
for i=1:N,
  TmpA=[]; TmpB=Rad(i,1);
  for j=1:n,
    TmpA=[TmpA,mod((TmpB + (j-1)*Angle + Tmp),TwoPi)];
  end
  Star=[Star;TmpA];
end
Wobble=sparse(N,N); jWobble=sparse(N,N); TmpN=size(Limbo,1);
for i=1:TmpN,
  Min=10; TmpA=Limbo(i,1); TmpB=Limbo(i,2); jMin=j;
  for j=1:n,
    Tmp=Star(TmpA,j)-Oclock(TmpA,TmpB);
    if(abs(Tmp)<abs(Min))
      Min=Tmp; jMin=j;
    end
  end
  Wobble(TmpA,TmpB)=Min; jWobble(TmpA,TmpB)=jMin;
end
Tmp=Angle/2;
for i=1:2:TmpN,
  TmpA=Limbo(i,1); TmpB=Limbo(i,2);
  if(abs(Wobble(TmpB,TmpA)) >= abs(Wobble(TmpA,TmpB)))
    TmpA=Limbo((i+1),1); TmpB=Limbo((i+1),2);
  end
```

```
  J=jWobble(TmpA,TmpB); v{1}=V{TmpA}{1,J};
  if(J==n)
    v{2}=V{TmpA}{1,1};
  else
    v{2}=V{TmpA}{1,(J+1)};
  end
  J=jWobble(TmpB,TmpA); v{3}=V{TmpB}{1,J};
  if(J==n)
    v{4}=V{TmpB}{1,1};
  else
    v{4}=V{TmpB}{1,(J+1)};
  end
  Max=0;
  if(Wobble(TmpA,TmpB)>=0)
    vMin=v{1};
  else
    vMin=v{2};
  end
  d1=R*cos(Tmp-abs(Wobble(TmpA,TmpB))); TmpD=(X(TmpA,1)-X(TmpB,1));
  a=(X(TmpA,2)-X(TmpB,2))/TmpD; b=(X(TmpA,1)*X(TmpB,2)-X(TmpB,1)*X(TmpA,2))/TmpD;
  a1=a; b1=vMin(1,2)-a1*vMin(1,1); x3=v{3}(1,1); y3=v{3}(1,2); x4=v{4}(1,1);
  y4=v{4}(1,2); TmpD=x3-x4; p=(y3-y4)/TmpD; q=(x3*y4-x4*y3)/TmpD; TmpD=a1-p;
  x=(q-b1)/TmpD; y=(a1*q-b1*p)/TmpD; dx=x-X(TmpB,1); dy=y-X(TmpB,2);
  r2=sqrt(dx*dx + dy*dy); d2=r2*cos(Tmp-abs(Wobble(TmpB,TmpA))); d=D(TmpA,TmpB);
  if((d1+d2)>=d)
    Ov(TmpA,TmpB)=1; Ov(TmpB,TmpA)=1; Ov1(TmpA,1)=1; Ov1(TmpB,1)=1;
  end
end
Clus=Ov;
for i=1:N,
  Clus(i,i)=1;
end
NClus=size(Clus,1); ClusA=Clus(1,:); NClusA=1;
for i=2:NClus,
  Joined=0;
  for j=1:NClusA,
    TmpC=Clus(i,:) | ClusA(j,:);
    if(sum(Clus(i,:) & ClusA(j,:)))
      ClusA(j,:)=TmpC; ClusB=ClusA; ClusA=TmpC; NClusB=NClusA;
      NClusA=1; Joined=1; break;
    end
  end
  if(~Joined)
    ClusA=[ClusA;Clus(i,:)]; NClusA=NClusA+1;
  else
    for j=1:NClusB,
      if(sum(ClusA(1,:) & ClusB(j,:)))
        ClusA(1,:)=ClusA(1,:) | ClusB(j,:);
      else
        ClusA=[ClusA;ClusB(j,:)]; NClusA=NClusA+1;
      end
    end
  end
end
Left=sparse(1,N); Right=sparse(1,N); Margin=0.1*Size;
for i=1:N,
  if(X(i,1)<=Margin)
    Left(1,i)=1;
  elseif(X(i,1)>=(Size-Margin))
    Right(1,i)=1;
  end
end
Plated=0;
for i=1:NClusA,
  if(sum(Left & ClusA(i,:)) & sum(Right & ClusA(i,:)))
    Plated=1; break;
  end
end
```

§ 7.6 Tilings

```
% tiling.m;  NB. must run one of the data below first.
qn=size(q,1); in2n=size(in2,1); in3n=size(in3,1); p=[];
p{1,1}=[o;o]'; p{1,2}=sz; r=[]; s=[];
for i=1:sz,
  r(1,i)=dx(m(i)); s(1,i)=dy(n(i));
end
p{1,3}=r'; p{1,4}=s'; iin=size(ii,1); Tmp=ones(iin,1);
TmpA=sparse(Tmp,ii(:,1),Tmp,1,sz); Tmp=[]; TmpB=[]; TmpC=[]; count=0;
for i=1:sz,
  if(~TmpA(i))
    count=count+1; Tmp=[Tmp;i,count]; TmpB=[TmpB;r(i)]; TmpC=[TmpC;s(i)];
  end
end
p{2,1}=Tmp; p{2,2}=count; p{2,3}=TmpB; p{2,4}=TmpC; iiin=size(iii,1);
Tmp=ones(iiin,1); TmpA=sparse(Tmp,iii(:,1),Tmp,1,sz);
Tmp=[]; TmpB=[]; TmpC=[]; count=0;
for i=1:sz,
  if(~TmpA(i))
    count=count+1; Tmp=[Tmp;i,count]; TmpB=[TmpB;r(i)]; TmpC=[TmpC;s(i)];
  end
end
p{3,1}=Tmp; p{3,2}=count; p{3,3}=TmpB; p{3,4}=TmpC; iv=[ii,2*ones(iin,1)];
for i=1:iiin,
  Tmp=0;
  for j=1:iin,
    if(iii(i,1)==ii(j,1))
      Tmp=1;
    end
  end
  if(~Tmp)
    iv=[iv;iii(i,:),3];
  end
end
ivn=size(iv,1); Tmp=ones(size(ivn,1),1);
TmpA=sparse(Tmp,iv(:,1),Tmp,1,sz); Tmp=[]; TmpB=[]; TmpC=[]; count=0;
for i=1:sz,
  if(~TmpA(i))
    count=count+1; Tmp=[Tmp;i,count]; TmpB=[TmpB;r(i)]; TmpC=[TmpC;s(i)];
  end
end
p{4,1}=Tmp; p{4,2}=count; p{4,3}=TmpB; p{4,4}=TmpC; map=[]; Tmp=ones(sz,1);
map{1,1}=sparse(p{1,1}(:,1),Tmp,p{1,1}(:,2),sz,1); count=p{1,2}; e=q;
v=[p{1,3},p{1,4}]; Tmp=ones(p{2,2},1);
for i=2:nx,
  map{i,1}=sparse(p{2,1}(:,1),Tmp,count*Tmp+p{2,1}(:,2),sz,1);
  for k=1:iin,
    map{i,1}(ii(k,1),1)=map{(i-1),1}(ii(k,2),1);
  end
  count=count+p{2,2}; % +iin;
  for k=1:qn,
    e=[e;map{i,1}(q(k,1)),map{i,1}(q(k,2))];
  end
  for k=1:in2n,
    e=[e;map{i,1}(in2(k,1)),map{(i-1),1}(in2(k,2))];
  end
  v=[v;(i-1)*dim1*Tmp+p{2,3},p{2,4}];
end
Tmp=ones(p{3,2},1);
for j=2:ny,
  map{1,j}=sparse(p{3,1}(:,1),Tmp,count*Tmp+p{3,1}(:,2),sz,1);
  for k=1:iiin,
    map{1,j}(iii(k,1),1)=map{1,(j-1)}(iii(k,2),1);
  end
  count=count+p{3,2}; % +iiin;
  for k=1:qn,
    e=[e;map{1,j}(q(k,1)),map{1,j}(q(k,2))];
  end
  for k=1:in3n,
    e=[e;map{1,j}(in3(k,1)),map{1,(j-1)}(in3(k,2))];
  end
  v=[v;p{3,3},(j-1)*dim2*Tmp+p{3,4}];
end
Tmp=ones(p{4,2},1);
for i=2:nx,
  for j=2:ny,
    map{i,j}=sparse(p{4,1}(:,1),Tmp,count*Tmp+p{4,1}(:,2),sz,1);
    for k=1:ivn,
      if(iv(k,3)==2)
        map{i,j}(iv(k,1),1)=map{(i-1),j}(iv(k,2),1);
```

```
      else
        map{i,j}(iv(k,1),1)=map{i,(j-1)}(iv(k,2),1);
      end
    end
    count=count+p{4,2}; % +ivn;
    for k=1:qn,
      e=[e;map{i,j}(q(k,1)),map{i,j}(q(k,2))];
    end
    for k=1:in2n,
      e=[e;map{i,j}(in2(k,1)),map{(i-1),j}(in2(k,2))];
    end
    for k=1:in3n,
      e=[e;map{i,j}(in3(k,1)),map{i,(j-1)}(in3(k,2))];
    end
    v=[v;(i-1)*dim1*Tmp+p{4,3},(j-1)*dim2*Tmp+p{4,4}];
  end
end
en=size(e,1); vn=size(v,1); cn=size(c,1);
for i=1:cn,
  c{i,2}=size(c{i},2);
end
C=[]; count=0;
for i=1:nx,
  for j=1:ny,
    for k=1:cn,
      count=count+1;
      C{count,1}=[];
      for m=1:c{k,2},
        C{count,1}=[C{count},map{i,j}(c{k,1}(m))];
      end
    end
  end
end
ciin=size(cii,1);
for i=1:ciin,
  cii{i,3}=size(cii{i,1},2); cii{i,4}=size(cii{i,2},2);
end
ciiin=size(ciii,1);
for i=1:ciiin,
  ciii{i,3}=size(ciii{i,1},2); ciii{i,4}=size(ciii{i,2},2);
end
for i=2:nx,
  for j=1:ciin,
    count=count+1; C{count,1}=[];
    for k=1:cii{j,3},
      C{count,1}=[C{count},map{i,1}(cii{j,1}(k))];
    end
    for k=1:cii{j,4},
      C{count,1}=[C{count},map{(i-1),1}(cii{j,2}(k))];
    end
  end
end
for i=2:ny,
  for j=1:ciiin,
    count=count+1;
    C{count,1}=[];
    for k=1:ciii{j,3},
      C{count,1}=[C{count},map{1,i}(ciii{j,1}(k))];
    end
    for k=1:ciii{j,4},
      C{count,1}=[C{count},map{1,(i-1)}(ciii{j,2}(k))];
    end
  end
end
civn=size(civ,1);
for i=1:civn,
  civ{i,5}=size(civ{i,1},2); civ{i,6}=size(civ{i,2},2);
  civ{i,7}=size(civ{i,3},2); civ{i,8}=size(civ{i,4},2);
end
for i=2:nx,
  for j=2:ny,
    for m=1:ciin,
      count=count+1; C{count,1}=[];
      for k=1:cii{m,3},
        C{count,1}=[C{count},map{i,j}(cii{m,1}(k))];
      end
      for k=1:cii{m,4},
        C{count,1}=[C{count},map{(i-1),j}(cii{m,2}(k))];
      end
    end
    for m=1:ciiin,
      count=count+1;
```

```
      C{count,1}=[];
      for k=1:ciii{m,3},
        C{count,1}=[C{count},map{i,j}(ciii{m,1}(k))];
      end
      for k=1:ciii{m,4},
        C{count,1}=[C{count},map{i,(j-1)}(ciii{m,2}(k))];
      end
    end
    for m=1:civn,
      count=count+1;
      C{count,1}=[];
      for k=1:civ{m,5},
        C{count,1}=[C{count},map{i,j}(civ{m,1}(k))];
      end
      for k=1:civ{m,6},
        C{count,1}=[C{count},map{(i-1),j}(civ{m,2}(k))];
      end
      for k=1:civ{m,7},
        C{count,1}=[C{count},map{(i-1),(j-1)}(civ{m,3}(k))];
      end
      for k=1:civ{m,8},
        C{count,1}=[C{count},map{i,(j-1)}(civ{m,4}(k))];
      end
    end
  end
end
Cn=size(C,1);
for i=1:Cn,
  C{i,2}=size(C{i,1},2);
end
x=[];
for i=1:Cn,
  Tmp=[0,0];
  for j=1:C{i,2},
    Tmp=Tmp+v(C{i,1}(j),:);
  end
  x=[x;Tmp/C{i,2}];
end
xn=size(x,1);
% for Cells
B=[]; Bxx=sparse(Cn,Cn); NeCMat=sparse(Cn,Cn); CVMat=sparse(Cn,vn);
for i=1:Cn,
  for j=1:C{i,2},
    CVMat(i,C{i,1}(j))=1;
  end
end
count=0;
for i=1:vn,
  Tmp=find(CVMat(:,i)); TmpN=size(Tmp,1);
  for j=1:(TmpN-1),
    for k=(j+1):TmpN,
      if(~NeCMat(Tmp(j),Tmp(k)))
        count=count+1; NeCMat(Tmp(j),Tmp(k))=1; NeCMat(Tmp(k),Tmp(j))=1;
        B=[B;Tmp(j),Tmp(k)]; Bxx(Tmp(j),Tmp(k))=count; Bxx(Tmp(k),Tmp(j))=count;
      end
    end
  end
end
Bn=count; A=x; N=size(A,1); LMat=sparse(1,N); UMat=sparse(1,N);
LB=min(x(:,1)); UB=max(x(:,1)); rng=UB-LB;
% LBc=.05*rng+LB;
LBc=.1*rng+LB; UBc=UB-LBc;
for i=1:N,
  if(A(i,1)<=LBc)
    LMat(1,i)=1;
  elseif(A(i,1)>=UBc)
    UMat(1,i)=1;
  end
end
NeMat=NeCMat; Blocked=randperm(Cn);
% for Bonds
NeBMat=sparse(Bn,Bn);
for i=1:Cn,
  [p,q,r]=find(Bxx(i,:)); nc=size(r,2);
  for j=1:(nc-1),
    for k=(j+1):nc,
      NeBMat(r(1,j),r(1,k))=1; NeBMat(r(1,k),r(1,j))=1;
    end
  end
end
A=B; N=size(A,1); LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:N,
```

```
  if((x(A(i,1),1)<=LBc) | (x(A(i,2),1)<=LBc))
    LMat(1,i)=1;
  elseif((x(A(i,1),1)>=UBc) | (x(A(i,2),1)>=UBc))
    UMat(1,i)=1;
  end
end
NeMat=NeBMat; Blocked=randperm(Bn);
% for cells
b=[]; bxx=sparse(Cn,Cn); NecMat=sparse(Cn,Cn); count=0;
for i=1:(Cn-1),
  for j=(i+1):Cn,
    if(sum(CVMat(i,:) & CVMat(j,:))==2)
      count=count+1; NecMat(i,j)=1; NecMat(j,i)=1; b=[b;i,j];
      bxx(i,j)=count; bxx(j,i)=count;
    end
  end
end
bn=count; A=x; N=size(A,1); LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:N,
  if(A(i,1)<=LBc)
    LMat(1,i)=1;
  elseif(A(i,1)>=UBc)
    UMat(1,i)=1;
  end
end
NeMat=NecMat; Blocked=randperm(Cn);
% for bonds
NebMat=sparse(bn,bn);
for i=1:Cn,
  [a,bb,c]=find(bxx(i,:)); nc=size(c,2);
  for j=1:(nc-1),
    for k=(j+1):nc,
      NebMat(c(1,j),c(1,k))=1; NebMat(c(1,k),c(1,j))=1;
    end
  end
end
A=b; N=size(A,1); LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:N,
  if((x(A(i,1),1)<=LBc) | (x(A(i,2),1)<=LBc))
    LMat(1,i)=1;
  elseif((x(A(i,1),1)>=UBc) | (x(A(i,2),1)>=UBc))
    UMat(1,i)=1;
  end
end
NeMat=NebMat; Blocked=randperm(bn);
% for vertices
NeVMat=sparse(vn,vn);
for i=1:en,
  NeVMat(e(i,1),e(i,2))=1; NeVMat(e(i,2),e(i,1))=1;
end
A=v; N=vn; LMat=sparse(1,N); UMat=sparse(1,N); LB=min(v(:,1));
UB=max(v(:,1)); rng=UB-LB; LBv=.05*rng+LB; UBv=UB-LBv;
for i=1:vn,
  if(v(i,1)<LBv)
    LMat(1,i)=1;
  end
  if(v(i,1)>UBv)
    UMat(1,i)=1;
  end
end
NeMat=NeVMat; Blocked=randperm(vn);
% for edges
EVMat=sparse(en,vn);
for i=1:en,
  EVMat(i,e(i,1))=1; EVMat(i,e(i,2))=1;
end
NeEMat=sparse(en,en);
for i=1:vn,
  Tmp=find(EVMat(:,i)); TmpN=size(Tmp,1);
  for j=1:(TmpN-1),
    for k=(j+1):TmpN,
      NeEMat(Tmp(j),Tmp(k))=1; NeEMat(Tmp(k),Tmp(j))=1;
    end
  end
end
A=e; N=en; LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:N,
  if((v(A(i,1),1)<=LBv) | (v(A(i,2),1)<=LBv))
    LMat(1,i)=1;
  elseif((v(A(i,1),1)>=UBv) | (v(A(i,2),1)>=UBv))
    UMat(1,i)=1;
  end
```

```
end
NeMat=NeEMat; Blocked=randperm(N);
% At the end of the day.
figure(1); clf; hold on;
for i=1:en,
  plot([v(e(i,1),1),v(e(i,2),1)],[v(e(i,1),2),v(e(i,2),2)]);
end
figure(2); clf; hold on;
for i=1:bn,
  plot([x(b(i,1),1),x(b(i,2),1)],[x(b(i,1),2),x(b(i,2),2)]);
end
figure(3); clf; hold on;
for i=1:Bn,
  plot([x(B(i,1),1),x(B(i,2),1)],[x(B(i,1),2),x(B(i,2),2)]);
end
% Here come data to be run first, though listed last.
% (1)  3_3[3^3]7_1[3^7]
clear all; sz=20; nx=7; ny=4; dx=0.5*[0;1;2;4;6;7;8;10;12]; Tmp=sqrt(3)/2;
dy=Tmp*[0;1;2;4;5;6;7;8;10;11;12]; dim1=max(dx); dim2=max(dy);
q=[1,2; 1,4; 2,3; 3,5; 4,7; 5,6; 5,8; 7,8; 7,9; 8,10; 9,12; 10,11; 10,14; 12,13;...
 13,14;,13,15; 14,16; 15,19; 16,17; 16,18; 18,20];
m=[3,4,6,1,7,9,3,5,2,7,8,2,3,5,1,7,9,6,3,4];
n=[1,1,2,3,3,3,4,4,5,6,6,7,8,8,9,9,9,10,11,11];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20];
ii=[4,6;15,17]; iii=[1,19;2,20]; in2=[9,11;12,11]; in3=[3,18];
c={[1,2,3,5,8,7,4];[7,8,10,14,13,12,9];[13,14,16,18,20,19,15]};
cii={[4,7,9],[11,10,8,5];[9,12],[11];[12,13,15],[16,14,10,11]};
ciii={[2,3],[18]}; civ={[1,4],[3,5],[18,16,17],[]};
% (2)  3_3[3^3]9_3[3^9]_II
clear all; sz=19; nx=4; ny=5; Tmp=1/sqrt(3); dx=Tmp*[0,1,2,3,4,5,6,7,8,9,10,11,12];
Tmp=1/3; dy=Tmp*[0,2,4,6,7,8,9,11,13,15,16,17,18]; dim1=max(dx); dim2=max(dy);
q=[1,3;2,4;3,5;3,6;4,7;4,9;5,6;5,9;6,8;7,10;8,11;9,12;10,13;12,15;...
 12,16;13,17;14,19;15,16;15,18;16,19;17,18];
m=[10,4,10,4,9,11,2,12,7,1,13,7,1,13,6,8,3,5,10];
n=[1,2,3,4,5,5,6,6,7,8,8,9,10,10,11,11,12,12,13];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19];
ii=[10,11;13,14]; iii=[1,19]; in2=[7,8]; in3=[2,17;2,18];
c={[3,6,5];[4,9,12,15,18,17,13,10,7];[12,16,15];[11,14,19,16,12,9,5,6,8]};
cii={[7,10],[8];}; ciii={[2],[17,18];[1,3,5,9,4,2],[18,15,16]};
civ={[2,4,7],[8,6,3,1],[14],[]};
% (3)  4_4[3^4]8_4[3^8]
clear all; sz=8; nx=10; ny=10; Tmp=1/sqrt(2); dx=[0,Tmp,1+Tmp,1+2*Tmp];
dy=[0,Tmp,1+Tmp,1+2*Tmp]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,3;2,4;3,5;5,7;6,8]; m=[2,3,1,4,1,4,2,3]; n=[1,1,2,2,3,3,4,4];
o=[1,2,3,4,5,6,7,8]; ii=[3,4;5,6]; iii=[1,7;2,8]; in2=[]; in3=[];
c={[1,2,4,6,8,7,5,3]}; cii={}; ciii={}; civ={[1,3],[2],[6],[]};
% (4)  3_3[3^3]8_2[3^8]
clear all; sz=25; nx=3; ny=6; dx=[0,1,2,3,4,5,6,7,8,9,10,11,12];
dy=.5*[0,2,4,5,6,8,10,11,12]; dim1=max(dx); dim2=max(dy);
q=[1,4;2,6;3,5;4,7;4,8;5,9;5,10;6,12;6,13;7,8;7,11;8,12;9,10;9,13;10,14;11,17;...
 12,15;13,16;15,19;15,20;16,21;16,22;17,23;18,25;19,20;19,23;20,24;21,22;21,24;22,25];
m=[3,7,11,3,11,7,2,4,10,12,1,5,9,13,5,9,1,13,4,6,8,10,3,7,11];
n=[1,1,1,2,2,3,4,4,4,4,5,5,5,5,6,6,7,7,8,8,8,8,9,9,9];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25];
ii=[11,14;17,18]; iii=[1,23;2,24;3,25]; in2=[]; in3=[];
c={[4,8,7];[5,10,9];[7,8,12,15,19,23,17,11];[6,13,16,21,24,20,15,12];...
 [9,10,14,18,25,22,16,13];[15,20,19];[16,22,21]}; cii={};
ciii={[2,6,12,8,4,1],[19,20];[3,5,9,13,6,2],[21,22]};
civ={[1,4,7,11],[10,5,3],[18],[]};
% (5)  3_3[3^3]9_3[3^9]_III
clear all; sz=12; nx=6; ny=4; Tmp=sqrt(3)/2; dx=Tmp*[0,1,2,3,4];
dy=.5*[0,3,5,6,9,11,12]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,3;2,3;3,4;4,5;4,6;5,7;6,8;7,9;9,11;10,12];
m=[2,4,3,3,2,4,1,5,1,5,2,4]; n=[1,1,2,3,4,4,5,5,6,6,7,7];
o=[1,2,3,4,5,6,7,8,9,10,11,12]; ii=[7,8;9,10]; iii=[1,11;2,12]; in2=[5,6]; in3=[];
c={[1,2,3];[4,6,8,10,12,11,9,7,5]}; cii={[5,7],[6]}; ciii={};
civ={[1,3,4,5],[6,4,3,2],[10],[]};
% (6)  4_3[3^4]10_6[3^10]_I
clear all; sz=20; nx=8; ny=4; dx=[0,2,3,5,6]; dy=3*[0,1,2,3,4];
dim1=max(dx); dim2=max(dy);
q=[1,2;1,6;2,3;2,7;3,4;4,5;6,7;6,9;7,10;9,10;10,11;11,12;11,14;12,13;12,15;14,15;14,18;15,19];
m=[1,2,3,4,5,1,2,5,1,2,3,4,5,3,4,1,2,3,4,5];
n=[1,1,1,1,1,2,2,2,3,3,3,3,3,4,4,5,5,5,5,5];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20];
ii=[1,5;6,8;9,13;16,20]; iii=[1,16;2,17;3,18;4,19;5,20]; in2=[]; in3=[];
c={[1,2,7,6];[2,3,4,5,8,13,12,11,10,7];[6,7,10,9];[11,12,15,14];[14,15,19,18]};
cii={[9,10,11,14,18,17,16],[19,15,12]}; ciii={}; civ={};
% (7)  3_3[3^3]9_3[3^9]_I
clear all; sz=37; nx=5; ny=3; x=2*sqrt(3)/(2+sqrt(3)); i=1/2; j=sqrt(3)/2;
dx=x*[0, i, i+j, 2*i+j, 3*i+j, 3*i+2*j, 4*i+2*j, 5*i+2*j, 5*i+3*j,...
 6*i+3*j, 7*i+3*j, 7*i+4*j, 8*i+4*j]; y=1/2/sqrt(3);
dy=x*[0, 2*y, 2*y+1, 4*y+1, 5*y+1, 5*y+1.5, 6*y+1.5, 8*y+1.5, 8*y+2.5, 10*y+2.5,...
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 11*y+2.5, 11*y+3, 12*y+3, 14*y+3, 14*y+4, 16*y+4, 17*y+4, 17*y+4.5, 18*y+4.5,...
 20*y+4.5, 20*y+5.5, 22*y+5.5, 23*y+5.5, 23*y+6, 24*y+6]; dim1=max(dx); dim2=max(dy);
q=[1,4;2,3;3,5;3,6;4,8;4,9;5,6;5,7;6,9;7,10;8,11;9,14;10,12;12,15;13,16;14,17; 14,19;...
 15,19;16,18;17,18;17,20;18,20;19,22;20,21;21,23;21,24;22,25;22,27;23,24; 23,26;...
 24,28;25,26;25,29;26,29;27,31;29,30;30,33;30,34;31,35;32,37;33,34;33,36;34,37;35,36];
m=[10,4,4,10,3,5,2,12,7,1,13,1,13,7,2,12,9,11,4,10,10,4,9,...
 11,6,8,1,13,7,7,1,13,6,8,3,5,10];
n=[1,2,3,4,5,5,6,6,7,8,8,9,9,10,11,11,12,12,13,14,15,16,17,...
 17,18,18,19,19,20,21,22,22,23,23,24,24,25];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,...
 24,25,26,27,28,29,30,31,32,33,34,35,36,37]; ii=[10,11;12,13;27,28;31,32];
iii=[1,37]; in2=[7,8;15,16]; in3=[2,35;2,36];
c={[3,6,5];[5,6,9,14,19,15,12,10,7];[4,8,11,13,16,18,17,14,9];...
 [14,17,20,21,23,26,25,22,19];[17,18,20];[21,24,23];[22,25,29,30,33,36,35,31,27];...
 [23,24,28,32,37,34,30,29,26];[25,26,29];[30,34,33]};
cii={[7,10],[8];[12,15],[16];[15,19,22,27],[24,21,20,18,16]};
ciii={[2],[35,36];[1,4,9,6,3,2],[36,33,34]}; civ={[2,3,5,7],[8,4,1],[32],[35]};
% (8)  4_2[3^4]10_4[3^10]
clear all; sz=42; nx=3; ny=6; i=sqrt(3)-1; j=1/2; k=sqrt(3);
dx=[0, i, k-j, k, k+j, k+i, 2*k-i, 2*k-j, 2*k, 2*k+j, 3*k-i, 3*k, 3*k+i,...
 4*k-j, 4*k, 4*k+j, 4*k+i, 5*k-i, 5*k-j, 5*k, 5*k+j, 6*k-i, 6*k];
i=(2/3)*(3-sqrt(3)); j=1/sqrt(3); k=1/2/sqrt(3);
dy=[0, i, 2-k, 2, 2+j, 3, 3+i, 5-k, 5, 5+j, 6]; dim1=max(dx); dim2=max(dy);
q=[1,11;2,5;3,6;5,7;5,8;6,9;6,10;7,12;7,15;8,12;8,17;9,13;9,18;10,13;...
 10,20;11,21;12,16;13,19;14,23;15,16;15,21;16,17;17,22;18,19;18,22;19,20;...
 20,23;21,24;22,31;23,25;24,26;24,27;25,28;25,29;26,30;26,33;27,30;27,35;...
 28,32;28,36;29,32;29,38;30,34;31,40;31,41;32,37;33,34;33,39;...
 34,35;35,40;36,37;36,41;37,38;38,42];
m=[1,9,15,23,9,15,8,10,14,16,1,9,15,23,6,9,11,13,15,18,4,12,20,4,...
 20,3,5,19,21,4,12,20,2,4,7,17,20,22,1,9,15,23];
n=[1,1,1,1,2,2,3,3,3,3,4,4,4,4,5,5,5,5,5,5,6,6,6,7,...
  7,8,8,8,8,9,9,9,10,10,10,10,10,10,11,11,11,11];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,...
 25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42];
ii=[1,4;11,14;39,42]; iii=[1,39;2,40;3,41;4,42]; in2=[]; in3=[];
c={[5,8,12,7];[6,10,13,9];[7,12,16,15];[8,17,16,12];[9,13,19,18];[10,20,19,13];...
 [15,16,17,22,31,40,35,27,24,21];[18,19,20,23,25,28,36,41,31,22];[24,27,30,26];...
 [25,29,32,28];[26,30,34,33];[27,35,34,30];[28,32,37,36];[29,38,37,32]};
cii={[11,21,24,26,33,39],[38,29,25,23]};
ciii={[2,5,7,15,21,11,1],[33,34,35];[3,6,9,18,22,17,8,5,2],[31];...
 [4,14,23,20,10,6,3],[36,37,38]}; civ={};
% (9)  4_3[3^4]10_6[3^10]_II
clear all; sz=56; nx=5; ny=3; i=2/(2-1/sqrt(3)); n=sqrt(3);
j=.5*(1.5*i-sqrt(3)); k=i/2; p=n-n*(1-i/n)-j; q=i;
dx=[0,j,k,p,q,n,n+j,n+k,n+p,n+q,2*n,2*n+j,2*n+k,2*n+p,2*n+q,3*n,3*n+j,...
 3*n+k,3*n+p,3*n+q,4*n]; m=3; j=i*(1/2+1/n)-1; tmp=1-i/sqrt(3);
p=i; k=p-tmp; q=p+tmp; r=q+i/2; s=2*i;
dy=[0,j,k,p,q,r,s,m,m+j,m+k,m+p,m+q,m+r,m+s,2*m,2*m+j,2*m+k,2*m+p,2*m+q,2*m+r,...
 2*m+s,3*m,3*m+j,3*m+k,3*m+p,3*m+q,3*m+r,3*m+s,4*m]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,7;2,8;3,6;5,6;6,10;7,8;7,12;8,13;10,11;10,14;11,18;11,19;12,13;13,16;...
 14,17;14,19;15,18;16,17;16,21;17,22;18,20;19,20;20,23;21,22;21,25;22,26;23,27;...
 24,28;25,26;25,28;26,29;27,30;27,31;28,32;29,30;29,33;30,34;32,35;33,34;33,38;...
 34,39;35,36;35,37;36,41;36,44;37,44;38,39;38,41;39,42;40,43;41,45;42,43;42,46;...
 43,47;44,45;45,48;46,47;46,51;47,52;48,49;48,50;49,55;50,54;51,52;51,55;52,56];
m=[1,5,15,21,9,13,1,5,21,13,17,1,5,11,21,6,10,20,14,18,6,10,18,2,6,10,16,...
  5,11,15,19,3,11,15,3,7,1,11,15,21,10,16,20,4,8,16,20,8,12,6,16,20,1,5,15,21];
n=[1,1,1,1,2,3,4,4,4,5,6,7,7,7,7,8,8,8,9,10,11,11,12,13,14,14,14,...
 15,15,15,16,17,18,18,19,20,21,21,21,21,22,22,22,23,24,25,25,26,27,28,28,28,29,29,29,29];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,...
 28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,56];
ii=[1,4;7,9;12,15;37,40;53,56]; iii=[1,53;2,54;3,55;4,56];
in2=[24,23;24,31;32,31]; in3=[5,49;5,50];
c={[1,2,8,7];[7,8,13,12];[10,11,19,14];[11,18,20,19];[14,19,20,23,27,30,29,26,22,17];...
 [16,17,22,21];[21,22,26,25];[25,26,29,33,38,41,36,35,32,28];[29,30,34,33];...
 [33,34,39,38];[35,36,44,37];[36,41,45,44];[38,39,42,46,51,55,49,48,45,41];...
 [42,43,47,46];[46,47,52,51]};
cii={[12,13,16,21,25,28,24],[23,20,18];[24],[31,27,23];[24,28,32],[31];...
 [32,35,37],[43,42,39,34,30,27,31];[37,44,45,48,50,54,53],[52,47,43]};
ciii={[5,6,10,14,17,16,13,8,2],[50];[5],[50,48,49];[5,6,3],[49];...
 [4,9,15,18,11,10,6,3],[51,52]};civ={};
% (10)  4_2[3^4]8_2[3^8]
clear all; sz=32; nx=4; ny=6; i=1/2; j=sqrt(3);
dx=[0,i,2*i,j,2*j,3*j-2*i,3*j-i,3*j,3*j+i,3*j+2*i,4*j,5*j,6*j-2*i,6*j-i,6*j];
i=(2*(sqrt(3)-1))/sqrt(3); j=(sqrt(3)/2)+i; k=(1/2/sqrt(3))+j; m=i+sqrt(3);
n=3; dy=[0,i,j,k,m,n,n+i,n+j,n+k,n+m,2*n]; dim1=max(dx); dim2=max(dy);
q=[1,5;2,10;3,11;5,7;6,8;7,9;7,14;8,12;8,15;9,13;10,17;10,18;11,18;11,19;13,14;...
 14,17;15,16;15,19;17,23;18,20;19,25;20,21;20,22;21,24;21,26;22,24;22,28;23,29;...
 23,30;24,27;25,31;25,32;26,27;26,30;27,28;28,31];
m=[1,5,11,15,1,15,2,14,1,5,11,15,1,3,13,15,4,8,12,8,7,9,4,8,12,6,8,10,1,5,11,15];
n=[1,1,1,1,2,2,3,3,4,4,4,4,5,5,5,5,6,6,6,7,8,8,9,9,9,10,10,10,11,11,11,11];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,...
 24,25,26,27,28,29,30,31,32];
```

```
ii=[1,4;5,6;9,12;13,16;29,32]; iii=[1,29;2,30;3,31;4,32]; in2=[]; in3=[];
c={[7,14,13,9];[8,12,16,15];[10,18,20,21,26,30,23,17];[11,19,25,31,28,22,20,18];...
 [20,22,24,21];[21,24,27,26];[22,28,27,24]}; cii={[5,7,9],[8];[13,14,17,23,29],[25,19,15]};
ciii={[2,10,17,14,7,5,1],[23];[3,11,18,10,2],[26,27,28]};
civ={[1],[6,8,15,19,11,3],[25],[]};
% (11)  4_3[3^4]8_3[3^8]_I
clear all; sz=22; nx=9; ny=3; m=sqrt(3); i=2/(2-1/m); n=m/2; tmp=(m-i)/2;
tmp1=i/4; j=n-tmp1; k=n-tmp; p=n+tmp; q=n+tmp1;
dx=[0,j,k,p,q,m,m+j,m+k,m+p,m+q,2*m]; m=2; n=1; tmp=.5*(1-i+sqrt(3)*i/2); tmp1=2*tmp;
j=m-tmp1; k=j+i/2; p=m+tmp; q=m+n-tmp; r=p+i/2; s=m+n+tmp1; u=2*m+n; t=u+n;
dy=[0,j,k,p,q,r,s,u,t,t+j,t+k,t+p,t+q,t+r,t+s,t+u,2*t]; dim1=max(dx); dim2=max(dy);
q=[1,3;3,5;4,6;5,7;5,8;6,7;7,9;8,9;9,10;10,11;10,12;11,13;13,16;14,15;15,17;...
 15,18;16,17;17,19;18,19;19,20;20,21;20,22];
m=[1,11,1,11,5,9,8,2,6,6,1,11,1,11,7,3,4,10,6,6,1,11];
n=[1,1,2,2,3,4,5,6,7,8,9,9,10,10,11,12,13,14,15,16,17,17];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22];
ii=[1,2;3,4;11,12;13,14;21,22]; iii=[1,21;2,22]; in2=[8,6;16,18]; in3=[];
c={[5,7,9,8];[10,12,14,15,17,16,13,11];[15,18,19,17]};
cii={[3,5,8],[6];[8,9,10,11],[10,9,7,6];[13,16],[18,15];[16,17,19,20,21],[20,19,18]};
ciii={[2,4,6,7,5,3,1],[20]}; civ={};
% (12)  4_3[3^4]8_3[3^8]_II
clear all; sz=41; nx=5; ny=3; dx=[0,1,2,3,4,5,6,7,8,9,10,11,12];
dy=2.5*[0,1,2,3,4,5,6,7,8]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,10;2,3;3,4;4,5;4,8;5,6;5,9;6,7;8,9;8,12;9,13;10,11;11,12;11,20;12,13;13,14;...
 14,15;14,17;15,16;15,18;17,18;17,22;18,23;19,20;19,25;20,21;21,22;21,30;22,23;23,24;...
 24,26;25,27;26,32;27,28;28,29;28,33;29,30;29,34;30,31;31,32;31,40;33,34;33,36;34,37];
m=[1,3,5,6,8,10,13,6,8,1,4,6,8,9,11,13,9,11,2,4,7,9,...
 11,12,2,12,2,3,5,7,10,12,3,5,1,3,5,6,8,10,13];
n=[1,1,1,1,1,1,1,2,2,3,3,3,3,3,3,3,4,4,5,5,5,5,5,5,6,6,7,7,7,7,7,7,8,8,9,9,9,9,9,9,9];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,...
 28,29,30,31,32,33,34,35,36,37,38,39,40,41]; ii=[1,7;10,16;35,41];
iii=[1,35;2,36;3,37;4,38;5,39;6,40;7,41]; in2=[19,24;25,26;27,32]; in3=[];
c={[1,2,3,4,8,12,11,10];[4,5,9,8];[5,6,7,16,15,14,13,9];[8,9,13,12];...
 [11,12,13,14,17,22,21,20];[14,15,18,17];[17,18,23,22];[19,20,21,30,29,28,27,25];...
 [21,22,23,24,26,32,31,30,21];[28,29,34,33];[33,34,37,36]};
cii={[10,11,20,19],[24,23,18,15,16];[19,25],[26,24];[25,27],[32,26]};
ciii={[3,4,5,6],[34,29,30,31]}; civ={[1],[6],[31,32],[27,28,33,36]};
% (13)  4_3[3^4]8_3[3^8]_III
clear all; sz=22; nx=10; ny=3; x=16*sqrt(3)/(4*(sqrt(3)+1)); i=x/4; j=2*sqrt(3);
dx=[0,i,2*i,j-2*i,j-i,j,j+i,j+2*i,2*j-2*i,2*j-i,2*j]; i=(sqrt(3)/4)*x; j=12;
dy=[0,x,x+i,x+2*i,j-2-x-2*i,j-2-x-i,j-2-x,j-2,j,j+x,j+x+i,j+x+2*i,2*j-2-x-2*i,...
 2*j-2-x-i,2*j-2-x,2*j-2,2*j]; dim1=max(dx); dim2=max(dy);
q=[1,3;3,7;4,5;5,6;6,7;7,8;8,9;9,10;10,11;10,12;11,13;13,15;14,17;15,16;16,17;17,18;...
 18,19;19,20;20,21;20,22]; m=[1,11,1,11,10,9,8,7,6,6,1,11,1,11,2,3,4,5,6,6,1,11];
n=[1,1,2,2,3,4,5,6,7,8,9,9,10,10,11,12,13,14,15,16,17,17];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22];
ii=[1,2;3,4;11,12;13,14;21,22]; iii=[1,21;2,22]; in2=[8,5;9,6;15,18;16,19]; in3=[];
c={[10,12,14,17,16,15,13,11]};
cii={[3,7,8],[5];[8,9],[6,5];[9,10,11],[10,9,8,7,6];[13,15],[17,18];[15,16],[18,19];,,,
 [16,17,18,19,20,21],[19,20]}; ciii={[2,4,5,6,7,3,1],[20]}; civ={}
% (14)  4_4[3^4]7_2[3^7]_II
clear all; sz=33; nx=6; ny=5; dx=[0,1,2,3,4,5,6,7,8,9,10,11,12];
dy=4*[0,1,2,3,4]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,8;2,3;3,4;4,5;4,10;5,6;5,11;6,7;8,9;9,10;9,16;10,11;11,12;12,13;12,18;...
 13,14;13,19;15,16;15,21;16,17;17,18;17,24;18,19;19,20;20,26;21,22;22,23;22,28;...
 23,24;23,29;24,25;25,26;25,32];
m=[1,3,5,6,8,10,13,1,4,6,8,9,11,13,2,4,7,9,11,12,2,3,5,7,10,12,1,3,5,6,8,10,13];
n=[1,1,1,1,1,1,1,2,2,2,2,2,2,2,3,3,3,3,3,3,4,4,4,4,4,4,5,5,5,5,5,5,5];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,...
 27,28,29,30,31,32,33]; ii=[1,7;8,14;27,33]; iii=[1,27;2,28;3,29;4,30;5,31;6,32;7,33];
in2=[15,20;21,26]; in3=[];
c={[1,2,3,4,10,9,8];[4,5,11,10];[5,6,7,14,13,12,11];[9,10,11,12,18,17,16];[12,13,19,18];...
 [15,16,17,24,23,22,21];[17,18,19,20,26,25,24];[22,23,29,28];[23,24,25,32,31,30,29]};
cii={[8,9,16,15],[20,19,13];[15,21],[26,20];[21,22,28,27],[32,25,26]}; ciii={}; civ={};
% (15)  3_3[3^3]12_6[3^12]
clear all; sz=17; nx=8; ny=5; i=2+sqrt(3); dx=[0,1,2,i,i+1,i+2,2*i];
i=sqrt(3); j=3+2*i; dy=[0,i,i+2,2*i+2,j,j+i,j+i+2,j+2*i+2,2*j]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,4;2,4;4,5;5,6;5,7;6,7;7,9;8,10;9,10;9,11;10,11;11,12;12,13;12,14;13,14;13,16;14,17];
m=[1,3,7,2,2,1,3,7,4,6,5,5,4,6,1,3,7]; n=[1,1,1,2,3,4,4,4,5,5,6,7,8,8,9,9,9];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17]; ii=[1,3;6,8;15,17]; iii=[1,15;2,16;3,17];
in2=[]; in3=[]; c={[1,2,4];[5,7,6];[9,10,11];[12,14,13]};
cii={[6,7,9,11,12,13,16],[17,14,12,11,10]}; ciii={};
civ={[1,4,5,6],[10,9,7,5,4,2],[13,14],[]};
% (16)  4_4[3^4]7_2[3^7]_I
clear all; sz=35; nx=6; ny=6; dx=[0,1,2,3,4,5,6]; dy=[0,1,2,3,4,5,6];
dim1=max(dx); dim2=max(dy);
q=[1,2;1,7;2,3;3,4;3,12;4,5;5,8;7,10;8,9;8,14;10,11;10,16;11,12;11,17;...
 12,13;13,14;13,18;14,19;16,17;17,22;18,19;18,23;19,20;20,21;20,28;22,23;...
 22,25;23,26;24,25;24,30;25,31;26,27;26,33;27,28;27,34;28,29];
m=[1,2,3,4,5,7,1,5,7,1,2,3,4,5,7,1,2,4,5,6,7,2,4,1,2,4,5,6,7,1,2,3,4,5,7];
n=[1,1,1,1,1,1,2,2,2,3,3,3,3,3,3,4,4,4,4,4,4,5,5,6,6,6,6,6,6,7,7,7,7,7,7];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,...
```

```
 27,28,29,30,31,32,33,34,35]; ii=[7,9;16,21;24,29];
iii=[1,30;2,31;4,33;5,34]; in2=[]; in3=[];
c={[1,2,3,12,11,10,7];[3,4,5,8,14,13,12];[13,14,19,18];[11,12,13,18,23,22,17];...
 [10,11,17,16];[18,19,20,28,27,26,23];[24,25,31,30];[26,27,34,33]};
cii={[16,17,22,25,24],[28,20];[10],[21,20,19,14,8,9]};
ciii={[2,3,4],[25,22,23,26];}; civ={[1],[9,8,5],[27,28],[24]};
% (17)  5_4[3^5]7_4[3^7]_I
clear all; sz=75; nx=2; ny=3; i=sqrt(3)/2;
dx=i*[0,1,2,3,4,6,7,8,9,10,12,13,14,15,16,18,19,20,21,22,24];
dy=.5*[0,1,2,4,6,7,8,10,12,13,14,16,18,19,20,22,24];
dim1=max(dx); dim2=max(dy);
q=[1,7;1,10;2,10;2,11;3,11;3,12;4,12;4,13;5,13;5,14;6,8;6,15;7,9;7,25;8,14;9,18;...
 10,19;11,20;12,16;13,17;14,23;16,17;16,21;17,22;18,27;19,25;19,28;20,26;20,29;...
 21,29;21,30;22,30;22,31;23,31;23,32;24,32;25,27;26,28;26,41;27,35;28,36;29,37;...
 30,38;31,33;32,34;33,34;33,39;34,40;35,43;35,44;36,44;36,45;37,41;37,46;38,42;...
 38,47;39,47;39,48;40,48;40,49;41,45;42,46;42,60;43,50;44,51;45,55;46,56;47,57;...
 48,58;50,51;50,53;51,54;53,62;54,62;54,63;55,63;55,64;56,64;56,65;57,60;57,66;...
 58,61;58,67;59,67;60,65;61,66;62,70;63,68;64,69;65,73;66,74;67,75;68,69;68,71;69,72];
m=[3,6,10,13,16,20,2,19,1,5,8,11,15,18,21,11,15,1,5,8,11,15,18,21,4,7,3,6,10,13,...
 16,20,16,20,3,6,10,13,16,20,9,12,1,5,8,11,15,18,21,1,5,21,1,5,8,11,15,18,21,14,...
 17,3,6,10,13,16,20,6,10,3,6,10,13,16,20];
n=[1,1,1,1,1,1,2,2,3,3,3,3,3,3,3,4,4,5,5,5,5,5,5,5,6,6,7,7,7,7,7,7,8,8,9,9,9,9,9,...
 9,10,10,11,11,11,11,11,11,11,11,12,12,12,13,13,13,13,13,13,13,14,14,15,15,15,15,15,...
 15,16,16,17,17,17,17,17,17];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,...
 30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,...
 56,57,58,59,60,61,62,63,64,65,66,67,68,69,70,71,72,73,74,75];
ii=[9,15;18,24;43,49;50,52;53,59]; iii=[1,70;2,71;3,72;4,73;5,74;6,75];
in2=[]; in3=[8,61];
c={[1,10,19,25,7];[2,11,20,26,28,19,10];[3,12,16,21,29,20,11];[4,13,17,16,12];...
 [5,14,23,31,22,17,13];[6,15,24,32,23,14,8];[7,25,27,18,9];[16,17,22,30,21];...
 [19,28,36,44,35,27,25];[20,29,37,41,26];[21,30,38,42,46,37,29];[22,31,33,39,47,38,30];...
 [23,32,34,33,31];[26,41,45,36,28];[33,34,40,48,39];[35,44,51,50,43];...
 [36,45,55,63,54,51,44];[37,46,56,64,55,45,41];[38,47,57,60,42];[39,48,58,61,66,57,47];...
 [40,49,52,59,67,58,48];[42,60,65,56,46];[50,51,54,62,53];[55,64,69,68,63]};
cii={[18,27,35,43],[40,34,32]};
ciii={[2,10,1],[62,54,63,68];[2,11,3],[69,68];[4,12,3],[69,64,56,65];...
 [5,13,4],[65,60,57,66];[5,14,8],[61,66];[8,6],[67,58,61]};
civ={[1,7,9],[6],[67,59],[62]};
% (18)  3_1[4^3]5_1[4^5]_I
clear all; sz=27; nx=4; ny=8; dx=[0,1,2,3,4,5,6,7,8]; dy=[0,1,2,3,4];
dim1=max(dx); dim2=max(dy);
q=[1,2;1,8;2,3;2,8;3,4;3,12;4,5;4,9;5,6;5,16;6,7;6,10;8,11;9,13;9,14;9,15;11,12;11,18;...
 12,13;12,23;13,14;13,19;14,15;14,19;15,16;15,19;16,17;16,25;18,21;18,22;19,24;20,26];
m=[1,2,3,5,7,8,9,1,5,9,1,3,4,5,6,7,9,1,5,9,1,2,3,5,7,8,9];
n=[1,1,1,1,1,1,1,2,2,2,3,3,3,3,3,3,3,4,4,4,5,5,5,5,5,5,5];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27];
ii=[1,7;8,10;11,17;18,20;21,27]; iii=[1,21;2,22;3,23;4,24;5,25;6,26;7,27]; in2=[]; in3=[];
c={[1,2,8];[2,3,12,11,8];[3,4,9,13,12];[4,5,16,15,9];[5,6,10,17,16];[6,7,10];...
 [9,14,13];[9,15,14];[11,12,23,22,18];[12,13,19,24,23];[13,14,19];[14,15,19];...
 [15,16,25,24,19];[16,17,20,26,25];[18,22,21]}; cii={}; ciii={[6,7],[20]}; civ={};
% (19)  3_1[4^3]5_1[4^5]_II
clear all; sz=13; nx=8; ny=8; dx=[0,1,2,3,4]; dy=[0,1,2,3,4]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,5;2,3;2,4;4,6;4,7;4,8;5,6;5,11;6,7;6,10;7,8;7,10;8,9;8,10;10,12];
m=[1,3,5,3,1,2,3,4,5,3,1,3,5]; n=[1,1,1,2,3,3,3,3,3,4,5,5,5];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13]; ii=[1,3;5,9;11,13]; iii=[1,11;2,12;3,13]; in2=[]; in3=[];
c={[1,2,4,6,5];[2,3,9,8,4];[4,7,6];[4,8,7];[5,6,10,12,11];[6,7,10];[7,8,10];[8,9,13,12,10]};
cii={}; ciii={}; civ={};
% (20)  5_3[3^5]8_6[3^8]_II
clear all; sz=110; nx=3; ny=2; i=sqrt(3)/2;
dx=[0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16];
dy=0.5*[0,1,2,4,6,7,8,10,12,13,14,16,18,19,20,22,24,25,26,28,30,31,32,34,...
 36,37,38,40,42,43,44,46,48]; dim1=max(dx); dim2=max(dy);
q=[1,5;1,8;2,6;2,9;3,9;3,10;4,10;4,11;5,7;5,19;6,8;7,14;8,15;9,12;10,13;12,13;...
 12,16;13,17;14,21;15,19;15,22;16,22;16,23;17,23;17,24;18,20;19,21;20,24;21,27;...
 22,25;23,26;24,30;25,26;25,28;26,29;27,31;27,34;28,34;28,35;29,35;29,36;30,32;...
 30,37;31,33;32,36;32,46;33,40;34,38;35,39;36,43;38,39;38,41;39,42;40,47;41,47;...
 41,48;42,48;42,49;43,45;43,50;44,46;45,49;45,58;46,50;47,51;48,52;49,55;50,56;...
 51,52;51,53;52,54;53,59;53,60;54,60;54,61;55,57;55,62;56,58;56,63;57,61;57,73;...
 58,62;59,64;60,65;61,69;62,70;64,65;64,67;65,68;67,74;68,74;68,75;69,72;69,76;...
 70,73;70,77;71,77;72,75;72,85;73,76;74,78;75,81;76,82;77,79;78,80;79,83;80,86;...
 80,87;81,84;81,88;82,85;82,89;83,89;83,90;84,87;84,100;85,88;86,91;87,95;88,96;...
 89,92;91,94;92,93;92,97;94,101;95,99;95,102;96,100;96,103;97,103;97,104;98,104;...
 99,101;100,102;101,107;102,108;103,105;104,106;105,106;105,109;106,110];
m=[3,7,11,15,2,6,1,5,9,13,17,9,13,1,5,9,13,17,4,16,3,7,11,15,7,11,3,7,...
 11,15,2,14,1,5,9,13,17,5,9,1,5,9,13,17,12,16,3,7,11,15,3,7,3,7,...
 11,15,10,14,1,5,9,13,17,1,5,17,1,5,9,13,17,8,12,3,7,11,15,3,15,3,...
 7,11,15,6,10,1,5,9,13,17,1,13,17,1,5,9,13,17,4,8,3,7,11,15,...
 11,15,3,7,11,15];
n=[1,1,1,1,2,2,3,3,3,3,3,4,4,5,5,5,5,5,6,6,7,7,7,7,8,8,9,...
 9,9,9,10,10,11,11,11,11,11,12,12,13,13,13,13,13,14,14,15,15,15,15,16,16,17,...
 17,17,17,18,18,19,19,19,19,19,20,20,20,21,21,21,21,21,22,22,23,23,23,23,24,24,...
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 25,25,25,25,26,26,27,27,27,27,27,28,28,28,29,29,29,29,29,30,30,31,31,31,...
 31,32,32,33,33,33,33];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,...
 28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52,...
 53,54,55,56,57,58,59,60,61,62,63,64,65,66,67,68,69,70,71,72,73,74,75,76,77,...
 78,79,80,81,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98,99,100,101,...
 102,103,104,105,106,107,108,109,110];
ii=[7,11;14,18;33,37;40,44;59,63;64,66;67,71;86,90;91,93;94,98];
iii=[1,107;2,108;3,109;4,110]; in2=[31,20;78,79]; in3=[6,99];
c={[1,8,15,19,5];[2,9,12,16,22,15,8,6];[3,10,13,12,9];[4,11,18,20,24,17,13,10];...
 [5,19,21,14,7];[12,13,17,23,16];[15,22,25,28,34,27,21,19];[16,23,26,25,22];...
 [17,24,30,32,36,29,26,23];[25,26,29,35,28];[27,34,38,41,47,40,33,31];...
 [28,35,39,38,34];[29,36,43,45,49,42,39,35];[30,37,44,46,32];[32,46,50,43,36];...
 [38,39,42,48,41];[41,48,52,51,47];[42,49,55,57,61,54,52,48];[43,50,56,58,45];...
 [45,58,62,55,49];[51,52,54,60,53];[53,60,65,64,59];[54,61,69,72,75,68,65,60];...
 [55,62,70,73,57];[57,73,76,69,61];[64,65,68,74,67];[68,75,81,84,87,80,78,74];...
 [69,76,82,85,72];[70,77,79,83,89,82,76,73];[72,85,88,81,75];...
 [80,87,95,99,101,94,91,86];[81,88,96,100,84];[82,89,92,97,103,96,88,85];...
 [83,90,93,92,89];[84,100,102,95,87];[92,93,98,104,97];[97,104,106,105,103]};
cii={[14,21,27,31],[20];[31,33],[30,24,20];[40,47,51,53,59],[56,50,46];...
 [67,74,78],[79,77];[64],[71,77,70,62,58,56,63];[78,80,86],[83,79]};
ciii={[1,8,6],[99,101];[2,6],[99,95,102];[2,9,3],[105,103,96,100,102];[3,10,4],[106,105]};
civ={[1,5,7],[4],[106,104,98],[101]};
% (21)  5_3[3^5]8_6[3^8]_III
clear all; sz=110; nx=4; ny=2; i=sqrt(3)/2;
dx=i*[0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16];
dy=0.5*[0,1,2,4,6,7,8,10,12,13,14,16,18,19,20,22,24,25,26,28,30,31,32,34,...
 36,37,38,40,42,43,44,46,48]; dim1=max(dx); dim2=max(dy);
q=[1,7;1,8;2,5;2,9;3,6;3,10;4,10;4,11;5,8;6,9;6,20;7,12;8,13;9,17;10,18;12,13;...
 12,15;13,16;15,22;16,22;16,23;17,23;17,24;18,20;18,25;19,21;20,24;21,25;22,28;...
 23,26;24,27;25,31;26,27;26,29;27,30;28,32;28,35;29,33;29,36;30,36;30,37;31,37;31,38;...
 32,34;33,35;33,47;34,39;35,43;36,44;37,40;39,42;40,41;40,45;42,49;43,49;43,50;44,47;...
 44,51;45,48;45,52;46,52;47,50;48,51;48,60;49,53;50,54;51,57;52,58;53,54;53,55;54,56;...
 55,59;55,62;56,62;56,63;57,63;57,64;58,60;58,65;59,61;59,73;60,64;61,68;62,69;63,66;...
 64,67;66,67;66,70;67,71;68,75;69,73;69,76;70,74;70,77;71,77;71,78;72,78;73,75;74,76;...
 74,85;75,79;76,82;77,83;78,80;79,81;80,84;81,87;81,88;82,88;82,89;83,85;83,90;84,86;...
 84,91;85,89;86,90;86,100;87,94;88,92;89,93;90,97;92,93;92,95;93,96;94,101;95,99;...
 95,102;96,102;96,103;97,103;97,104;98,100;99,101;100,104;101,107;102,108;103,105;...
 104,106;105,106;105,109;106,110];
m=[3,7,11,15,6,10,1,5,9,13,17,1,5,17,1,5,9,13,17,12,16,3,7,11,15,7,11,3,7,11,15,2,...
 6,1,5,9,13,17,1,13,17,1,5,9,13,17,8,12,3,7,11,15,3,7,3,7,11,15,2,14,1,5,9,13,17,9,...
 13,1,5,9,13,17,4,8,3,7,11,15,3,15,3,7,11,15,10,14,1,5,9,13,17,5,9,1,5,9,13,17,4,...
 16,3,7,11,15,11,15,3,7,11,15];
n=[1,1,1,1,2,2,3,3,3,3,3,4,4,4,5,5,5,5,5,6,6,7,7,7,7,8,8,9,9,9,9,10,10,11,11,11,...
 11,11,12,12,12,13,13,13,13,13,13,14,14,15,15,15,15,16,16,17,17,17,17,18,18,19,19,19...
 ,19,19,20,20,21,21,21,21,21,22,22,23,23,23,23,24,24,25,25,25,25,26,26,27,27,27,...
 27,27,28,28,29,29,29,29,29,30,30,31,31,31,31,32,32,33,33,33,33];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,...
 31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,56,57,58,...
 59,60,61,62,63,64,65,66,67,68,69,70,71,72,73,74,75,76,77,78,79,80,81,82,83,84,85,86,...
 87,88,89,90,91,92,93,94,95,96,97,98,99,100,101,102,103,104,105,106,107,108,109,110];
ii=[7,11;12,14;15,19;34,38;39,41;42,46;61,65;68,72;87,91;94,98];
iii=[1,107;2,108;3,109;4,110]; in2=[32,21;79,80]; in3=[5,99];
c={[1,8,13,12,7];[2,9,17,23,16,13,8,5];[3,10,18,20,6];[6,20,24,17,9];...
 [12,13,16,22,15];[16,23,26,29,33,35,28,22];[17,24,27,26,23];[18,25,31,37,30,27,24,20];...
 [26,27,30,36,29];[28,35,43,49,42,39,34,32];[29,36,44,47,33];[30,37,40,45,48,51,44,36];...
 [31,38,41,40,37];[33,47,50,43,35];[40,41,46,52,45];[43,50,54,53,49];...
 [44,51,57,63,56,54,50,47];[45,52,58,60,48];[48,60,64,57,51];[53,54,56,62,55];...
 [55,62,69,73,59];[56,63,66,70,74,76,69,62];[57,64,67,66,63];[58,65,72,78,71,67,64,60];...
 [59,73,75,68,61];[66,67,71,77,70];[69,76,82,88,81,79,75,73];[70,77,83,85,74];...
 [71,78,80,84,86,90,83,77];[74,85,89,82,76];[81,88,92,95,99,101,94,87];...
 [82,89,93,92,88];[83,90,97,103,96,93,89,85];[84,91,98,100,86];[86,100,104,97,90];...
 [92,93,96,102,95];[97,104,106,105,103]};
cii={[15,22,28,32],[21];[12],[19,21,25,18,10,4,11];[32,34],[31,25,21];...
 [42,49,53,55,59,61],[58,52];[68,75,79],[80,78];[79,81,87],[84,80]};
ciii={[1,8,5],[99,101];[5,2],[102,95,99];[2,9,6,3],[105,103,96,102];[3,10,4],[106,105]};
civ={[1,7],[4],[106,104,100,98],[101]};
% (22)  5_2[3^5]12_12[3^12]
clear all; sz=34; nx=5; ny=3; i=sqrt(3)/2; dx=i*[0,1,2,3,4,5,6,7,8];
dy=[0,.5,1,3,3.5,4,5,6,6.5,7,9,9.5,10,11,12]; dim1=max(dx); dim2=max(dy);
q=[1,3;1,6;2,4;2,6;3,5;3,11;4,7;4,12;5,8;6,9;8,13;9,11;9,12;10,14;11,13;12,14;...
 13,15;14,16;15,17;16,18;17,19;17,21;18,20;18,23;19,22;19,27;20,22;20,28;21,24;...
 22,25;24,27;25,29;25,30;26,28;27,29;28,30;29,31;30,32;31,32;31,33;32,34];
m=[3,7,2,8,1,5,9,1,5,9,4,6,3,7,3,7,3,7,4,6,1,5,9,1,5,9,2,8,3,7,3,7,3,7];
n=[1,1,2,2,3,3,3,4,4,4,5,5,6,6,7,7,8,8,9,9,10,10,10,11,11,11,12,12,13,13,14,14,15,15];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,...
 28,29,30,31,32,33,34]; ii=[5,7;8,10;21,23;24,26]; iii=[1,33;2,34]; in2=[15,16]; in3=[];
c={[1,6,9,11,3];[2,4,12,9,6];[3,11,13,8,5];[4,7,10,14,12];...
 [9,12,14,16,18,20,22,19,17,15,13,11];[17,19,27,24,21];[18,23,26,28,20];...
 [19,22,25,29,27];[20,28,30,25,22];[25,30,32,31,29]};
cii={[8,13,15],[16,14];[15,17,21],[18,16]}; ciii={[1,6,2],[32,31]};
civ={[1,3,5],[4,2],[32,30,28,26],[27,29,31]};
```

```
% (23)  5_4[3^5]7_4[3^7]_II
clear all; sz=21; nx=3; ny=5; i=sqrt(3)/2; dx=i*[0,1,2,3,4,6,7,8,9,10,12];
dy=[0,.5,1,3,3.5,4,6]; dim1=max(dx); dim2=max(dy);
q=[1,4;1,6;2,5;2,7;3,8;3,9;4,7;4,14;5,8;6,10;7,11;8,12;10,14;11,16;11,17;12,15;...
 12,17;13,18;14,16;15,18;16,19;17,20;18,21];
m=[3,6,10,4,7,1,5,8,11,1,5,8,11,2,9,3,6,10,3,6,10];
n=[1,1,1,2,2,3,3,3,3,4,4,4,4,5,5,6,6,6,7,7,7];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21];
ii=[6,9;10,13]; iii=[1,19;2,20;3,21]; in2=[]; in3=[5,15];
c={[1,4,14,10,6];[2,5,8,12,17,11,7];[3,9,13,18,15,12,8];[4,7,11,16,14]};
cii={}; ciii={[1,4,7,2],[17,11,16];[2,5],[15,12,17];[5,8,3],[18,15]};
civ={[1,6],[3],[18,13],[14,16]};
% (24)  5_3[3^5]8_6[3^8]_I
clear all; sz=30; nx=8; ny=6; i=sqrt(3)/2; dx=[0,1,2,3,4,5,6,7,8];
dy=[0,.5,1,3,3.5,4,6,6.5,7,9,9.5,10,12]; dim1=max(dx); dim2=max(dy);
q=[1,3;1,6;2,4;2,7;3,5;3,11;4,6;5,8;6,9;8,13;9,11;9,14;10,12;11,13;12,14;13,15;...
 14,16;15,17;15,20;16,18;16,21;17,19;18,20;18,26;19,22;20,23;21,24;22,27;23,25;...
 23,28;24,26;25,27;26,28;27,29;28,30];
m=[3,7,2,6,1,5,9,1,5,9,4,8,3,7,3,7,2,6,1,5,9,1,5,9,4,8,3,7,3,7];
n=[1,1,2,2,3,3,3,4,4,4,5,5,6,6,7,7,8,8,9,9,9,10,10,10,11,11,12,12,13,13];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30];
ii=[5,7;8,10;19,21;22,24]; iii=[1,29;2,30]; in2=[17,12]; in3=[4,25];
c={[1,6,9,11,3];[2,7,10,12,14,9,6,4];[3,11,13,8,5];[9,14,16,18,20,15,13,11];...
 [15,20,23,25,27,22,19,17];[16,21,24,26,18];[18,26,28,23,20]};
cii={[8,13,15,17],[12];[17,19],[16,14,12]};
ciii={[1,6,4],[25,27];[4,2],[28,23,25]}; civ={[1,3,5],[2],[28,26,24],[27]};
% (25)  4_2[3^4]12_6[3^12]
clear all; sz=21; nx=10; ny=6; i=sqrt(3)/2;
dx=[0,1-i,.5,1,1.5,1+i,2,1.5+i,2+i,2.5+i,2+2*i]; i=1/sqrt(3); j=sqrt(3)/6;
d=sqrt(3)+1.5; dy=[0,i,i+j,3*i,3*i+1,d,d+i,d+i+j,d+3*i,d+3*i+1,2*d];
dim1=max(dx); dim2=max(dy);
q=[1,2;1,6;2,3;2,5;3,7;5,6;5,7;6,8;7,8;8,9;9,10;9,11;11,12;11,14;12,13;13,14;...
 13,15;14,16;15,16;16,17;17,20;17,21];
m=[1,4,7,11,4,3,5,4,4,2,6,9,9,8,10,9,9,1,4,7,11];
n=[1,1,1,1,2,3,3,4,5,6,6,6,7,8,8,9,10,11,11,11,11];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21];
ii=[1,4;18,21]; iii=[1,18;2,19;3,20;4,21]; in2=[10,12;10,15]; in3=[];
c={[1,2,5,6];[2,3,7,5];[5,7,8,6];[11,12,13,14];[13,15,16,14]};
cii={[10],[15,13,12]}; ciii={};
civ={[1,6,8,9,10],[12,11,9,8,7,3],[17],[18];[3,2],[4],[17,16,15],[10,9,11,14,16,17,20]};
% (26)  4_2[3^4]18_12[3^18]
clear all; sz=36; nx=6; ny=3; i=sqrt(3); j=6/i; d=2*j+i;
dx=[0,i,j,d-j,d-i,d,d+i,d+j,2*d-j,2*d-i,2*d];
dy=[0,3,4,6,7,9,10,13,15,18,19,21,22,24,25,28,30]; dim1=max(dx); dim2=max(dy);
q=[1,3;2,4;3,5;3,8;4,6;4,9;5,7;7,8;8,11;9,10;9,13;11,12;11,15;12,13;12,14;13,16;...
 14,15;14,16;15,17;16,17;17,18;18,19;18,20;19,21;19,22;20,21;20,24;21,23;22,23;...
 22,26;23,24;24,27;25,26;25,29;26,31;27,28;27,32;29,31;30,32;31,33;32,34;33,35;34,36];
m=[1,11,2,10,1,11,1,3,9,11,4,6,8,6,5,7,6,6,5,7,6,4,6,8,1,3,9,11,1,11,2,10,1,11,1,11];
n=[1,1,2,2,3,3,4,4,4,4,5,5,5,6,7,7,8,9,10,10,11,12,12,12,...
 13,13,13,13,14,14,15,15,16,16,17,17];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,...
 25,26,27,28,29,30,31,32,33,34,35,36];
ii=[1,2;5,6;7,10;25,28;29,30;33,34;35,36]; iii=[1,35;2,36]; in2=[]; in3=[];
c={[3,8,7,5];[4,6,10,9];[11,12,14,15];[12,13,16,14];[14,16,17,15];[18,20,21,19];...
 [19,21,23,22];[20,24,23,21];[25,26,31,29];[27,28,30,32]};
cii={[1,3,5],[4];[7,8,11,15,17,18,19,22,26,25],[27,24,20,18,17,16,13,9];[29,31,33],[32]};
ciii={[1,3,8,11,12,13,9,4,2],[34,32,27,24,23,22,26,31,33]}; civ={};
% (27)  5_3[3^5]7_3[3^7]_I
clear all; sz=60; nx=4; ny=2; x=sqrt(3); dx=x*[0,1,2,3,4,5,6];
dy=[0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18]; dim1=max(dx); dim2=max(dy);
q=[1,5;2,5;2,6;3,6;3,7;4,7;5,8;6,11;7,9;8,10;9,12;10,13;10,14;11,14;11,15;12,15;...
 12,16;13,17;14,18;15,22;17,18;17,20;18,21;20,24;21,24;21,25;22,25;22,26;23,26;...
 24,27;25,28;26,31;27,28;27,29;28,30;29,32;29,33;30,33;30,34;31,34;31,35;32,38;...
 33,36;34,37;36,37;36,39;37,40;38,42;39,42;39,43;40,43;40,44;41,44;42,47;43,45;...
 44,46;45,46;45,48;46,49;47,50;47,51;48,51;48,52;49,52;49,53;50,54;...
 51,58;52,55;54,57;55,56;55,59];
m=[1,3,5,7,2,4,6,2,6,2,4,6,1,3,5,7,1,3,7,1,3,5,7,2,4,6,2,4,2,4,6,1,3,5,7,3,...
 5,1,3,5,7,2,4,6,4,6,2,4,6,1,3,5,7,1,5,7,1,3,5,7];
n=[1,1,1,1,2,2,2,3,3,4,4,4,5,5,5,5,6,6,6,7,7,7,7,8,8,8,9,9,10,10,10,11,11,11,...
 11,12,12,13,13,13,13,14,14,14,15,15,16,16,16,17,17,17,17,18,18,18,19,19,19,19];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,...
 31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,56,57,58,59,60];
ii=[1,4;13,16;17,19;20,23;32,35;38,41;50,53;54,56;57,60];
iii=[1,57;2,58;3,59;4,60]; in2=[8,9]; in3=[];
c={[2,6,11,14,10,8,5];[3,7,9,12,15,11,6];[10,14,18,17,13];[11,15,22,25,21,18,14];...
 [17,18,21,24,20];[21,25,28,27,24];[22,26,31,34,30,28,25];[27,28,30,33,29];...
 [29,33,36,39,42,38,32];[30,34,37,36,33];[31,35,41,44,40,37,34];[36,37,40,43,39];...
 [39,43,45,48,51,47,42];[40,44,46,45,43];[45,46,49,52,48];[49,53,56,55,52]};
cii={[1,5,8],[9,7];[8,10,13],[12,9];[17],[23,26,22,15,12,16];...
 [20,24,27,29,32],[31,26];[38,42,47,50],[49,46,44]};
ciii={[1,5,2],[51,47,50,54];[2,6,3],[55,52,48,51];[3,7,4],[56,55]}; civ={};
% (28)  5_3[3^5]7_3[3^7]_II
```

```
clear all; sz=32; nx=5; ny=3; nx=3; ny=2; x=sqrt(3)/2; dx=[0,2,3,4,6,7,8];
dy=0.5*[0,2,4,5,6,10,11,12,14,16,17,18,22,23,24]; dim1=max(dx); dim2=max(dy);
q=[1,4;2,8;4,5;4,7;5,10;7,11;8,10;8,12;9,12;10,11;11,13;12,14;13,16;13,17;14,15;
 14,17;15,18;15,20;16,21;17,19;19,20;19,22;20,24;21,25;22,25;22,26;23,24;24,26;
 25,27;26,28;27,29;27,30;28,31;28,32;29,31];
m=[1,4,7,1,2,7,1,4,7,3,2,5,2,5,6,1,4,7,4,5,1,4,7,6,2,5,2,5,3,1,4,7];
n=[1,1,1,2,2,2,3,3,3,4,5,5,6,6,7,8,8,8,9,9,10,10,10,11,12,12,13,13,14,15,15,15];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32];
ii=[1,3;4,6;7,9;16,18;21,23;30,32]; iii=[1,30;2,31;3,32]; in2=[]; in3=[5,29];
c={[4,5,10,11,7];[8,12,14,17,13,11,10];[13,17,19,22,25,21,16];[14,15,20,19,17];...
 [15,18,23,24,20];[19,20,24,26,22];[22,26,28,31,29,27,25]};
cii={[7,11,13,16],[15,14,12];[21,25,27,30],[28,26,24]};
ciii={[1,4,5],[29,27];[2,8,10,5],[29];[2,8,12,9,6,3],[28]}; civ={[1],[],[28,26,24,23],[25,27]};
% (29)  3_2[4^3]5_2[4^5]_I
clear all; sz=15; nx=6; ny=6; dx=[0,1,2,3,4]; dy=[0,1,2,3,4]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,6;2,3;2,5;3,4;3,5;5,7;5,8;6,7;6,10;7,8;7,10;8,9;8,13;10,12;11,14];
m=[1,3,4,5,3,1,2,3,5,1,5,1,3,4,5]; n=[1,1,1,1,2,3,3,3,3,4,4,5,5,5,5];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15]; ii=[1,4;6,9;10,11;12,15]; iii=[1,12;2,13;3,14;4,15];
in2=[]; in3=[];
c={[1,2,5,7,6];[2,3,5];[3,4,9,8,5];[5,8,7];[6,7,10];[7,8,13,12,10];[8,9,11,14,13]};
cii={}; ciii={}; civ={[1],[3],[11],[]};
% (30)  3_2[4^3]5_2[4^5]_II
clear all; sz=41; nx=3; ny=3; dx=[0,1,2,3,4,5,6,7,8]; dy=[0,1,2,3,4,5,6,7,8];
dim1=max(dx); dim2=max(dy);
q=[1,6;2,3;2,8;3,4;3,13;4,5;4,9;5,9;6,7;6,15;7,8;7,11;8,11;8,12;9,10;9,14;11,12;...
 11,16;12,13;12,22;13,14;13,17;14,17;14,18;15,16;15,20;16,20;16,21;17,18;17,23;...
 18,19;18,27;20,21;21,22;21,29;22,23;22,25;23,25;23,26;24,28;25,26;25,30;26,27;26,35;...
 27,28;27,31;28,31;28,32;29,30;29,33;30,33;30,34;31,32;31,36;32,40;33,34;33,37;34,35;...
 35,36;35,38;36,38;36,39];
m=[1,5,6,8,9,1,3,4,8,9,3,4,6,7,1,2,6,7,9,1,2,4,5,9,4,5,7,8,2,3,7,8,2,3,5,6,1,5,6,8,9];
n=[1,1,1,1,1,2,2,2,2,2,3,3,3,3,4,4,4,4,5,5,5,5,5,6,6,6,6,7,7,7,7,8,8,8,8,9,9,9,9,9];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,...
 29,30,31,32,33,34,35,36,37,38,39,40,41];
ii=[1,5;6,10;15,19;20,24;37,41]; iii=[1,37;2,38;3,39;4,40;5,41]; in2=[29,32]; in3=[7,34];
c={[2,3,13,12,8];[3,4,9,14,13];[4,5,9];[5,10,9];[6,7,11,16,15];[7,8,11];[8,12,11];...
 [9,10,19,18,14];[11,12,22,21,16];[12,13,17,23,22];[13,14,17];[14,18,17];[15,16,20];...
 [16,21,20];[17,18,27,26,23];[18,19,24,28,27];[21,22,25,30,29];[22,23,25];[23,26,25];...
 [25,26,35,34,30];[26,27,31,36,35];[27,28,31];[28,32,31];[29,30,33];[30,34,33];...
 [31,32,40,39,36];[35,36,38];[36,39,38]};
cii={[20,21,29],[32,28];[29,33,37],[40,32]}; ciii={[1,6,7],[34,33];[7,8,2],[35,34]}; civ={};
% (31)  3_3[4^3]5_3[4^5]
clear all; sz=9; nx=6; ny=7; dx=[0,2,3,4,6]; dy=[0,2.5,5]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,6;2,3;2,5;3,4;3,5;5,7;5,8]; m=[1,2,4,5,3,1,2,4,5]; n=[1,1,1,1,2,3,3,3,3];
o=[1,2,3,4,5,6,7,8,9]; ii=[1,4;6,9]; iii=[2,7;3,8;4,9]; in2=[]; in3=[];
c={[1,2,5,7,6];[2,3,5];[5,8,7]}; cii={[1,6],[8,5,3]}; ciii={}; civ={};
% (32)  3_1[4^3]6_2[4^6]_I
clear all; sz=21; nx=3; ny=3; dx=[0,2,3,5,7,8,10];
dy=[0,2,3,5,7,8,10]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,7;2,3;2,7;3,4;3,6;4,5;4,8;6,10;6,11;6,12;7,9;9,10;9,15;10,11;10,14;11,12;...
 11,14;12,13;12,14;14,19;15,17;15,18;16,20];
m=[1,3,4,5,7,4,1,7,1,2,4,6,7,4,1,7,1,3,4,5,7]; n=[1,1,1,1,1,2,3,3,4,4,4,4,4,6,5,5,7,7,7,7,7];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21]; ii=[1,5;7,8;9,13;15,16;17,21];
iii=[1,17;2,18;3,19;4,20;5,21]; in2=[]; in3=[];
c={[1,2,7];[2,3,6,10,9,7];[3,4,8,13,12,6];[4,5,8];[6,11,10];[6,12,11];[9,10,14,19,18,15];...
 [10,11,14];[11,12,14];[12,13,16,20,19,14];[15,18,17]}; cii={[15,17],[20]}; ciii={}; civ={};
% (33)  3_1[4^3]6_2[4^6]_II
clear all; sz=12; nx=6; ny=7; dx=[0,1,3,5,6]; dy=[0,2,3,5]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,8;2,3;2,6;3,4;3,6;4,5;4,6;6,7;7,9;7,10;7,11]; m=[1,2,3,4,5,3,3,1,2,3,4,5];
n=[1,1,1,1,1,2,3,4,4,4,4,4]; o=[1,2,3,4,5,6,7,8,9,10,11,12]; ii=[1,5;8,12];
iii=[1,8;2,9;3,10;4,11;5,12]; in2=[]; in3=[];
c={[1,2,6,7,9,8];[2,3,6];[3,4,6];[7,10,9];[7,11,10]}; cii={[1,8],[11,7,6,4]}; ciii={}; civ={};
% (34)  3_2[4^3]6_4[4^6]
clear all; sz=7; nx=10; ny=7; dx=[0,1,2,4]; dy=[0,3,6]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,6;2,3;2,4;4,5;4,6]; m=[1,3,4,2,1,3,4]; n=[1,1,1,2,3,3,3]; o=[1,2,3,4,5,6,7];
ii=[5,7;1,3]; iii=[1,5;2,6;3,7]; in2=[]; in3=[]; c={[1,2,4];[4,5,6]};
cii={[1,4,5],[6,4,2]}; ciii={}; civ={};
% (35)  3_3[4^3]6_6[4^6]
clear all; sz=11; nx=8; ny=5; dx=[0,1,2,3,4]; y=sqrt(3); dy=y*[0,1,2,3,4]; dim1=max(dx);
dim2=max(dy); q=[1,2;2,3;2,4;3,4;4,6;4,7;5,6;5,8;6,7;6,8;8,9;8,10];
m=[1,3,5,4,1,3,5,2,1,3,5]; n=[1,1,1,2,3,3,3,4,5,5,5]; o=[1,2,3,4,5,6,7,8,9,10,11];
ii=[1,3;5,7;9,11]; iii=[1,9;2,10;3,11]; in2=[]; in3=[]; c={[2,3,4];[4,7,6];[5,6,8];[8,10,9]};
cii={[1,2,4,6,5],[4];[5,8,9],[10,8,6]}; ciii={}; civ={};
% (36)  3_1[4^3]8_4[4^8]
clear all; sz=12; nx=7; ny=7; dx=[0,2,3,5]; dy=[0,2,3,5]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,5;2,3;2,5;3,4;3,6;5,7;7,9;7,10;8,11]; m=[1,2,3,4,1,4,1,4,1,2,3,4];
n=[1,1,1,1,2,2,3,3,4,4,4,4]; o=[1,2,3,4,5,6,7,8,9,10,11,12]; ii=[1,4;5,6;7,8;9,12];
iii=[1,9;2,10;3,11;4,12]; in2=[]; in3=[]; c={[1,2,5];[2,3,6,8,11,10,7,5];[3,4,6];[7,10,9]};
cii={[7,9],[11]}; ciii={}; civ={};
% (37)  3_1[5^3]4_2[5^4]_I
clear all; sz=8; nx=16; ny=4; dx=[0,1,2]; y=sqrt(3); dy=[0,2,y+2,y+4,2*y+4]; dim1=max(dx);
dim2=max(dy); q=[1,2;1,3;3,4;3,5;4,5;5,6;6,7;6,8]; m=[1,3,1,3,2,2,1,3]; n=[1,1,2,2,3,4,5,5];
o=[1,2,3,4,5,6,7,8]; ii=[1,2;3,4;7,8]; iii=[1,7;2,8]; in2=[5,5;6,6]; in3=[];
```

```
c={[1,2,4,3];[3,4,5];[6,8,7]}; cii={[3,5,6,7],[6,5]}; ciii={}; civ={};
% (38)  3_1[5^3]4_2[5^4]
clear all; sz=25; nx=3; ny=3; dx=[0,1,2,3,4]; dy=[0,1,2,3,4]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,6;2,3;2,6;2,7;3,4;3,7;3,8;4,5;4,9;6,7;6,11;7,8;7,12;8,9;8,12;8,13;9,10;9,13;...
 9,14;11,12;11,16;12,13;12,17;13,14;13,18;14,15;14,18;14,19;15,19;16,17;16,21;17,18;...
 17,21;17,22;18,19;18,23;19,20;19,24;20,24];
m=[1,2,3,4,5,1,2,3,4,5,1,2,3,4,5,1,2,3,4,5];
n=[1,1,1,1,1,2,2,2,2,2,3,3,3,3,3,4,4,4,4,5,5,5,5,5];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25];
ii=[1,5;6,10;11,15;16,20;21,25]; iii=[1,21;2,22;3,23;4,24;5,25]; in2=[]; in3=[];
c={[1,2,6];[2,7,6];[2,3,7];[3,8,7];[3,4,9,8];[4,5,10,9];[6,7,12,11];[7,8,12];[8,13,12];...
 [8,9,13];[9,14,13];[9,10,15,14];[11,12,17,16];[12,13,18,17];[13,14,18];[14,19,18];...
 [14,15,19];[15,20,19];[16,17,21];[17,22,21];[17,18,23,22];[18,19,24,23];[19,20,24]};
cii={[16,21],[24]}; ciii={}; civ={};
% (39)  3_2[5^3]4_4[5^4]
clear all; sz=12; nx=4; ny=4; t=37*pi/180; y=cos(t/2)/tan(t/2);
dx=[0,1,2,y+1,y+2,y+3,2*y+2]; dy=[0,1,y,y+1,y+2,2*y+1,2*y+2]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,5;2,4;2,5;3,4;4,6;4,7;5,6;5,8;6,7;6,8;6,9;7,9;8,10;8,11;9,11;9,12];
m=[1,3,7,5,2,4,6,2,5,1,3,7]; n=[1,1,1,2,3,4,4,5,6,7,7,7]; o=[1,2,3,4,5,6,7,8,9,10,11,12];
ii=[1,3;10,12]; iii=[1,10;2,11;3,12]; in2=[5,7;8,7]; in3=[4,9];
c={[1,2,5];[2,4,6,5];[4,7,6];[5,6,8];[6,7,9];[6,9,11,8]};
cii={[1,5],[7,4];[5,8],[7];[8,10],[9,7];[8,11],[12]}; ciii={[2,4],[9];[4,3],[9]}; civ={};
```

§ 7.7 Covering lattices

```
% cover contour
clear all; St=sum(100*clock); rand('state',St); CaN=100; X=rand(CaN,2);
[Va,Ca]=voronoin(X); VaN=size(Va,1); Vin=sparse(VaN,1);
for i=1:VaN,
  if((Va(i,1)<=1) & (Va(i,1)>=0) & (Va(i,2)<=1) & (Va(i,2)>=0))
    Vin(i,1)=1;
  end
end
C0=[]; Xn=[]; count=0;
for i=1:CaN,
  TmpN=size(Ca{i},2); Tmp=1;
  for j=1:TmpN,
    if(~Vin(Ca{i}(1,j),1))
      Tmp=0; break;
    end
  end
  if(Tmp)
    count=count+1; C0{count,1}=TmpN; C0{count,2}=Ca{i}; Xn=[Xn;i];
  end
end
CN=size(C0,1); C1=[];
for i=1:CN,
  Tmp=[C0{i,2},C0{i,2}(1,1)];
  for j=1:C0{i,1},
    C1{i,j}(1,1)=(Va(Tmp(1,j),1)+Va(Tmp(1,(j+1)),1))/2;
    C1{i,j}(1,2)=(Va(Tmp(1,j),2)+Va(Tmp(1,(j+1)),2))/2;
  end
end
C{1}=C1; figure(1); clf; hold on;
for i=1:CN,
  x=[]; y=[];
  for j=1:C0{i,1},
    x=[x,C{1}{i,j}(1,1)]; y=[y,C{1}{i,j}(1,2)];
  end
  x=[x,x(1,1)]; y=[y,y(1,1)]; plot(x,y);
end
axis equal;
n=8;
for k=2:n,
  Cn=[];
  for i=1:CN,
    TmpX=[]; TmpY=[];
    for j=1:C0{i,1},
      TmpX=[TmpX,C{k-1}{i,j}(1,1)]; TmpY=[TmpY,C{k-1}{i,j}(1,2)];
    end
    TmpX=[TmpX,TmpX(1,1)]; TmpY=[TmpY,TmpY(1,1)];
    for j=1:C0{i,1},
      Cn{i,j}(1,1)=(TmpX(1,j)+TmpX(1,(j+1)))/2;
      Cn{i,j}(1,2)=(TmpY(1,j)+TmpY(1,(j+1)))/2;
    end
  end
  C{k}=Cn;
end
figure(2); clf; hold on;
for i=1:CN,
```

```
  x=[]; y=[];
  for j=1:CO{i,1},
    x=[x,C{n}{i,j}(1,1)]; y=[y,C{n}{i,j}(1,2)];
  end
  x=[x,x(1,1)]; y=[y,y(1,1)]; plot(x,y);
end
axis equal; DEVV=sparse(VaN,VaN);
DVO=[];
for i=1:CN,
  TmpN=CO{i,1}; Tmp=[CO{i,2},CO{i,2}(1,1)];
  for j=1:TmpN,
    V1=Tmp(1,j); V2=Tmp(1,(j+1));
    if(~DEVV(V1,V2))
      dx=Va(V2,1)-Va(V1,1); dy=Va(V2,2)-Va(V1,2);
      TmpA=sqrt(dx*dx + dy*dy); DEVV(V1,V2)=TmpA; DEVV(V2,V1)=TmpA;
    end
    DVO{i,1}{1,1}(1,j)=DEVV(V1,V2); dx=Va(V1,1)-X(Xn(i,1),1);
    dy=Va(V1,2)-X(Xn(i,1),2); TmpA=sqrt(dx*dx + dy*dy); DVO{i,1}{2,1}(1,j)=TmpA;
  end
end
AO=[];
for i=1:CN,
  TmpN=CO{i,1}; Tmp=[DVO{i,1}{2,1},DVO{i,1}{2,1}(1,1)]; At=0;
  for j=1:TmpN,
    a=DVO{i,1}{1,1}(1,j); b=Tmp(1,j); c=Tmp(1,(j+1)); s=(a+b+c)/2;
    Ai=sqrt(s*(s-a)*(s-b)*(s-c)); At=At+Ai;
  end
  AO=[AO;At];
end
DV=[]; A=[];
for i=1:n,
  TmpA=[];
  for j=1:CN,
    TmpN=CO{j,1}; TmpB=[];
    for k=1:TmpN,
      TmpB=[TmpB;C{i}{j,k}];
    end
    TmpB=[TmpB;TmpB(1,:)];
    for k=1:TmpN,
      x1=TmpB(k,1); y1=TmpB(k,2); x2=TmpB((k+1),1); y2=TmpB((k+1),2);
      dx=x2-x1; dy=y2-y1; TmpA{j,1}(1,k)=sqrt(dx*dx + dy*dy);
      dx=x1-X(Xn(j,1),1); dy=y1-X(Xn(j,1),2); TmpA{j,2}(1,k)=sqrt(dx*dx+dy*dy);
    end
  end
  DV{i}=TmpA; TmpA=[];
  for j=1:CN,
    TmpN=CO{j,1}; TmpB=[DV{i}{j,2},DV{i}{j,2}(1,1)]; At=0;
    for k=1:TmpN,
      a=DV{i}{j,1}(1,k); b=TmpB(1,k); c=TmpB(1,(k+1)); s=(a+b+c)/2;
      Ai=sqrt(s*(s-a)*(s-b)*(s-c)); At=At+Ai;
    end
    TmpA=[TmpA;At];
  end
  A{i}=TmpA;
end
p=[sum(AO)];
for i=1:n,
  p=[p,sum(A{i})];
end
p=p*100/p(1); figure(3); clf; plot(0:n,p);
```

§ 7.8 TEX's macros

```
% thshead.tex
\def\Ordinate{\ifnum\day>30 1 \else\ifnum\day>20 \day-20 \else\day\fi\fi}
\def\date{{\number\day\/{\ifcase\Ordinate\or $^{st}$\or $^{nd}$\or $^{rd}$
 \else $^{th}$\fi}}
  {\ifcase\month\or January\or February\or March\or April\or May\or June\or July\or
    August\or September\or October\or November\else December\fi}, {\number\year}}
\def\dat[#1:#2:#3]{\begingroup\tmp=#1 \ifnum\tmp>30 \tmp=1 \else\ifnum\tmp>20
 \advance\tmp-20 \fi\fi
 {#1\/{\ifcase\tmp\or $^{st}$\or $^{nd}$\or $^{rd}$ \else $^{th}$\fi}} #2 #3
 \endgroup}
\input manmac % \tracingall
\hsize=6in  \vsize=9.70820393249937in  \maxdepth=2pt \parindent=2pc\pagewidth=\hsize
\pageheight=\vsize
\input epsf
\def\home{/home/mjkpjkt2}\def\dry#1{\ifcase#1\or
 \home/ar/thes1\or \home/ts\or \home/tiyapan/Extremum/DsgnExReport\or \home/xfg\or
 \home/ths\or \home/tiyapan/Perco/Papers\or \home/tiyapan/Extremum/Matlab\or
 \home/ar/rc1\or \home/ar/trans\or\home/ar/wn21\or \home/ar/wn23\or \home/ar/wn7\or
 \home/phy\fi}
\def\o[#1:#2]{$#1^{\hbox{\sevenrm #2}}$} \def\Title{UMIST Thesis by K N Tiyapan}
\font\tenbm=cmmib10 \font\ninett=cmtt9 \font\sixit=cmti6 \font\twelverm=cmr12
\def\[#1]{{\it #1}} \def\(#1){{\bf #1}} \def\r[#1]{{\rm #1}} \def\s[#1]{{\sl #1}}
\def\beginsection#1\par{\vskip\z@ plus.3\vsize\penalty-250
 \vskip\z@ plus-.3\vsize\bigskip\vskip\parskip
 \message{#1}\leftline{\bf#1}\nobreak\smallskip\noindent}
\def\ifundefined#1{\expandafter\ifx\csname#1\endcsname\relax}
\def\ix[#1](#2){\expandafter\def\csname#1\endcsname{#2}}
\def\x[#1]{\ifundefined{#1}\ix[#1]($\aleph$)\fi \csname#1\endcsname} \input ref
\newwrite\ref \immediate\openout\ref=ref
\newdimen\dtmp \newdimen\dtmpi \newdimen\dtmpii
\newcount\tmp
\def\Bf#1:{{\bf #1}} \def\ct{\hfil\par} \def\cut{\par\vfill\break}\def\der[#1]{{\bf #1},}
\def\It#1:{{\it#1}} \def\pc{\hbox{Pc}}      % percolation
\def\Sl#1:{{\sl#1}}
\def\w#1:{\dtmpi=\hsize \advance\dtmpi-.3em \dtmpii=\dtmpi \advance\dtmpii-.3em
  \parshape=2 0pt \dtmpi .3em \dtmpii {\tenpoint\bf #1.~}}
\def\vws[#1]#2:#3:{{\bf #1} ({\sl #2} in {\sl #3})}  % explain vowel sounds
\def\Lift#1#2{\raise#1em\hbox{#2}} \def\beginindex{\begingroup
 \parindent=1em \maxdepth=\maxdimen
 \def\par{\endgraf \futurelet\next\inxentry}
 \obeylines \everypar={\hangindent 2\parindent}
 \exhyphenpenalty=10000 \raggedright}
\def\inxentry{\ifx\next\sub \let\next=\subentry
 \else\ifx\next\endindex \let\next=\vfill
 \else\let\next=\mainentry \fi\fi \next}
\def\endindex{\mark{}\break\endgroup}
\let\sub=\indent \newtoks\maintoks \newtoks\subtoks
\def\mainentry#1,{\mark{}\noindent
 \maintoks={#1}\mark{\the\maintoks}#1,}
\def\subentry\sub#1,{\mark{\the\maintoks}\indent
 \subtoks={#1}\mark{\the\maintoks\sub\the\subtoks}#1,}
\def\boxit#1{\vbox{\hrule\hbox{\vrule\kern3pt
  \vbox{\kern3pt#1\kern3pt}\kern3pt\vrule}\hrule}}
\def\bxt<#1>{\vbox{\hrule\hbox{\vrule\kern.7pt
  \vbox{\kern.7pt#1\kern.7pt}\kern.7pt\vrule}\hrule}}
\def\square#1#2{{\vcenter{\vbox{\hrule height.#2pt
  \hbox{\vrule width.#2pt height#1pt \kern#1pt
    \vrule width.#2pt}
  \hrule height.#2pt}}}}
\def\sqr{\mathchoice\square34\square34\square{2.1}3\square{1.5}3}
\def\Le{\;{\raise.3em\hbox{$<$}}{\kern-.8em\lower.3em\hbox{$=$}}\;}
\def\Ge{\;{\raise.3em\hbox{$>$}}{\kern-.8em\lower.3em\hbox{$=$}}\;}
\def\matrix#1{\null\,\vcenter{\normalbaselines\m@th
    \ialign{\hfil$##$\hfil&&\quad\hfil$##$\hfil\crcr
      \mathstrut\crcr\noalign{\kern-\baselineskip}
      #1\crcr\mathstrut\crcr\noalign{\kern-\baselineskip}}}\,}
\def\pmatrix#1{\left(\matrix{#1}\right)}
\def\Gl{\mathop{\raise.23em\hbox{$>$}\kern-.71em\lower.23em\hbox{$<$}}}
\def\Mod{\mathop{\rm mod}}
\def\Lift#1#2{\raise#1em\hbox{#2}}
\def\uncatcodespecials{\def\do##1{\catcode`##1=12 }\dospecials}
\newcount\lineno
\def\setupverbatim{\tt \lineno=0
  \def\par{\leavevmode\endgraf} \catcode`\`=\active
  \obeylines \uncatcodespecials \obeyspaces
  \everypar{\advance\lineno by1 \llap{\sevenrm\the\lineno\ \ }}}
{\catcode`\`=\active \gdef|{\char'174}\gdef`{\relax\lq}\obeyspaces\global\let =\ }
\def\listing#1{\par\smallskip\begingroup\setupverbatim\ninett\baselineskip.9em\input#1
  \endgroup}
```

```
\def\tran#1{\begingroup\ninepoint\baselineskip.8em\parindent=1em\input#1\endgroup}
\def\picx#1:#2:{\epsfxsize=#1 \epsffile{#2.eps}}
\def\picy#1:#2:{\epsfysize=#1 \epsffile{#2.eps}}
\def\nin#1:{\begingroup\baselineskip1em\leftskip1em\parindent-1em % internet info
  \def\net##1:##2:##3:{{\tt ##1}{\it ##2}##3\par}\input #1\endgroup}
\def\bib#1:{\begingroup\baselineskip1em\leftskip1em\parindent-1em\input #1\endgroup}
\def\art#1:#2:#3:#4:{{\bf #1}~#2~{\it #3} #4\par}
\def\bbib{\begingroup\baselineskip1em\leftskip1em\parindent-1em}
\def\ebib{\endgroup}
% \pictri <file1>:<file2>:<file3>:<num1>:<num2>:<num3>:
\def\pictri#1:#2:#3:#4:#5:#6:{\smallskip
 \centerline{\vbox{\dtmp=\hsize \advance\dtmp-1em
  \dtmpi=\hsize \advance\dtmpi-2em \divide\dtmpi by3
  \halign to\dtmp{\hfil##\hfil&\hfil##\hfil&\hfil##\hfil\cr
   \epsfxsize=\dtmpi \epsffile{#1.eps} &\epsfxsize=\dtmpi \epsffile{#2.eps} &
   \epsfxsize=\dtmpi \epsffile{#3.eps} \cr
   (#4) &(#5) &(#6)\cr}}}}
\def\pictry#1:#2:#3:#4:#5:#6:{\smallskip
 \centerline{\vbox{\dtmp=\hsize \advance\dtmp-1em
  \dtmpi=\hsize \advance\dtmpi-2em \divide\dtmpi by3
  \halign to\dtmp{\hfil##\hfil&\hfil##\hfil&\hfil##\hfil\cr
   \epsfysize=\dtmpi \epsffile{#1.eps} &\epsfysize=\dtmpi \epsffile{#2.eps} &
   \epsfysize=\dtmpi \epsffile{#3.eps} \cr
   (#4) &(#5) &(#6)\cr}}}}
\def\picsept#1:#2:#3:#4:#5:#6:#7:#8:{\smallskip
 \centerline{\vbox{\dtmp=\hsize \advance\dtmp-1em
  \dtmpi=\hsize \advance\dtmpi-4em \divide\dtmpi by7
  \halign to\dtmp{\hfil##\hfil&\hfil##\hfil&\hfil##\hfil&\hfil##\hfil&\hfil##\hfil&
    \hfil##\hfil&\hfil##\hfil\cr
   \epsfxsize=\dtmpi \epsffile{#8#1.eps} &
   \epsfxsize=\dtmpi \epsffile{#8#2.eps} &
   \epsfxsize=\dtmpi \epsffile{#8#3.eps} &
   \epsfxsize=\dtmpi \epsffile{#8#4.eps} &
   \epsfxsize=\dtmpi \epsffile{#8#5.eps} &
   \epsfxsize=\dtmpi \epsffile{#8#6.eps} &
   \epsfxsize=\dtmpi \epsffile{#8#7.eps} \cr}}}}
\def\picsepu#1:#2:#3:#4:{\smallskip
 \centerline{\vbox{\dtmp=\hsize \advance\dtmp-1em
  \dtmpi=\hsize \advance\dtmpi-4em \divide\dtmpi by7
  \halign to\dtmp{##\hfil&\hfil##\hfil&\hfil##\hfil&\hfil##\cr
   \epsfxsize=\dtmpi \epsffile{#4#1.eps} &
   \epsfxsize=\dtmpi \epsffile{#4#2.eps} &
   \epsfxsize=\dtmpi \epsffile{#4#3.eps} \cr}}}}
% \pclr <dim_x>:<filename>:<directory>:<index>:<caption>:
\def\pclr#1:#2:#3:#4:#5:{\dtmp=\hsize \advance\dtmp-1em \dtmpi=\dtmp
  \advance\dtmpi-#1 \smallskip
 \def\pcl{\centerline{\vbox{\hsize=\dtmp \hbox{\epsfxsize=#1 \epsffile{#2.eps}\hfil
  \vbox{\hsize=\dtmpi\noindent{\bf Figure \x[#4]} #5\smallskip}}}}}
 \def\pcr{\centerline{\vbox{\hsize=\dtmp
  \hbox{\vbox{\hsize=\dtmpi\noindent{\bf Figure \x[#4]} #5\smallskip} \hfil
  \epsfxsize=#1 \epsffile{#3#2.eps}}}}}
 \ifodd\pageno \pcr \else \pcl \fi\smallskip}
% \pplr <dim_x>:<filename>:<directory>:<index>:<caption>:<paragraph>:
\def\pplr#1:#2:#3:#4:#5:#6:{\dtmp=\hsize \advance\dtmp-1em \dtmpi=\dtmp
  \advance\dtmpi-#1 \smallskip
 \def\pcl{\centerline{\vbox{\hsize=\dtmp \hbox{\epsfxsize=#1 \epsffile{#3#2.eps}\hfil
  \vbox{\hsize=\dtmpi\parindent=0pt #6\hfil\smallskip\noindent
   {\bf Figure \x[#4]} #5\smallskip}}}}}
 \def\pcr{\centerline{\vbox{\hsize=\dtmp
  \hbox{\vbox{\hsize=\dtmpi\noindent#6\hfil\smallskip
   \noindent{\bf Figure \x[#4]} #5\smallskip} \hfil
  \hbox{\epsfxsize=#1 \epsffile{#3#2.eps}}}}}}
 \ifodd\pageno \pcr \else \pcl \fi\smallskip}
\def\plr#1:#2:#3:#4:#5:#6:{\begingroup\dtmp=\hsize \advance\dtmp-1em \dtmpi=\dtmp
 \advance\dtmpi-#1 \smallskip
 \def\pcl{\centerline{\vbox{\hsize=\dtmp \hbox{\epsfxsize=#1 \epsffile{#3#2.eps}\hfil
  \vbox{\hsize=\dtmpi\noindent{#6\hfil\smallskip\bf #4.} #5\smallskip}}}}}
 \def\pcr{\centerline{\vbox{\hsize=\dtmp
  \hbox{\vbox{\hsize=\dtmpi\noindent#6\hfil\smallskip{\bf #4.} #5\smallskip} \hfil
  \hbox{\epsfxsize=#1 \epsffile{#3#2.eps}}}}}}
 \ifodd\pageno \pcr \else \pcl \fi\smallskip\endgroup}
\def\plrs#1:#2:#3:#4:#5:#6:{\begingroup\dtmp=\hsize \advance\dtmp-1em \dtmpi=\dtmp
 \advance\dtmpi-#1 \smallskip
 \def\pcl{\centerline{\vbox{\hsize=\dtmp \hbox{\epsfxsize=#1 \epsffile{#3#2.ps}\hfil
  \vbox{\hsize=\dtmpi\noindent{#6\hfil\smallskip\bf #4.} #5\smallskip}}}}}
 \def\pcr{\centerline{\vbox{\hsize=\dtmp
  \hbox{\vbox{\hsize=\dtmpi\noindent#6\hfil\smallskip{\bf #4.} #5\smallskip} \hfil
  \hbox{\epsfxsize=#1 \epsffile{#3#2.ps}}}}}}
 \ifodd\pageno \pcr \else \pcl \fi\smallskip\endgroup}
% \twop:<xsize1>:<file1>:<label1>:<xsize2>:<file2>:<label2>:
\def\twop:#1:#2:#3:#4:#5:#6:{
 \centerline{\hfil\epsfxsize=#1 \epsffile{#2.eps}\hfil\epsfxsize=#4
```

```
  \epsffile{#5.eps}\hfil}
 \centerline{\hfil #3\hfil\hfil\hfil #6\hfil}}
\def\tps:#1:#2:#3:#4:#5:#6:{
 \centerline{\hfil\epsfxsize=#1 \epsffile{#2.ps}\hfil\epsfxsize=#4
  \epsffile{#5.ps}\hfil}
 \centerline{\hfil #3\hfil\hfil\hfil #6\hfil}}
% \ppr<directory>:<fig1>:<fig1d>:<fig2>:<fig2d>{<capt1>}{<capt2>}
\def\ppr#1:#2:#3:#4:#5:#6#7{\begingroup\dtmp=\hsize \advance\dtmp-3em \dtmpi=\dtmp
  \divide\dtmpi by 4 \smallskip\parindent=0pt \dtmpii=0pt
 \setbox1=\vbox{\hsize=2\dtmpi #6} \setbox2=\vbox{\hsize=2\dtmpi #7}
 \ifdim\dtmpii<\ht1 \dtmpii=\ht1\fi \ifdim\dtmpii<\ht2 \dtmpii=\ht2\fi
 \centerline{\epsfxsize=\dtmpi \epsffile{#1#2.eps}~\epsfxsize=\dtmpi
  \epsffile{#1#3.eps}\hfil
  \epsfxsize=\dtmpi \epsffile{#1#4.eps}~\epsfxsize=\dtmpi \epsffile{#1#5.eps}}\smallskip
 \centerline{\vbox to\dtmpii{\hsize=\dtmpi\unvbox1\vfil}\hfil
  \vbox to\dtmpii{\hsize=\dtmpi\unvbox2\vfil}}
 \endgroup}
\def\prlr#1:#2:#3:#4{\begingroup\dtmp=\hsize \advance\dtmp-3em \dtmpi=\dtmp
   \divide\dtmpi by 4
  \smallskip\parindent=0pt
 \setbox1=\vbox{\hsize=2\dtmpi \epsfxsize=\dtmpi \epsffile{#1#2.eps}\hfil
  \epsfxsize=\dtmpi \epsffile{#1#3.eps}}
 \setbox2=\vbox{\hsize=2\dtmpi #4}
 \def\pcl{\centerline{\copy1\hfil\copy2}} \def\pcr{\centerline{\copy2\hfil\copy1}}
 \ifodd\pageno \pcr \else \pcl \fi\endgroup}
\def\ptlr#1:#2:#3:#4:#5{\begingroup\dtmp=\hsize \advance\dtmp-3em \dtmpi=\dtmp
   \divide\dtmpi by 4
  \smallskip\parindent=0pt
 \setbox1=\vbox{\hsize=3\dtmpi \epsfxsize=\dtmpi \epsffile{#1#2.eps}\hfil
  \epsfxsize=\dtmpi \epsffile{#1#3.eps}\hfil \epsfxsize=\dtmpi \epsffile{#1#4.eps}}
 \setbox2=\vbox{\hsize=\dtmpi #5}
 \def\pcl{\centerline{\copy1\hfil\copy2}} \def\pcr{\centerline{\copy2\hfil\copy1}}
 \ifodd\pageno \pcr \else \pcl \fi\endgroup}
\def\capt#1:#2:#3:{\smallskip
 \centerline{\dtmp=\hsize \advance\dtmp-2em \vbox{\hsize \dtmp \noindent
 {\bf #1 #2} {\it #3}}}\smallskip}
\def\balg{\begingroup\smallskip\begingroup\obeylines \sfcode`;=3000
 \def\b[##1]{{\bf ##1}} \def\i[##1]{{\it ##1}} \def\r[##1]{{\rm ##1}}\endgroup}
\def\ealg{\par\endgroup\smallskip}
\def\q{\quad} \def\qq{\qquad} \def\qqq{\qq\q} \def\qqqq{\qq\qq} \def\qqqqq{\qq\qq\q}
 \def\qqqqqq{\qq\qq\qq}
\def\picl#1:#2:#3:#4:{\dtmp=\hsize \advance\dtmp-1em \dtmpi=\dtmp
   \advance\dtmpi-#1 \smallskip
 \centerline{\vbox{\halign to\dtmp{##\hfil&\hfil##\cr
 \epsfxsize=#1 \epsffile{#2.eps}&\vbox{\hsize=\dtmpi
   \noindent{\bf Figure \x[#3]} #4\smallskip}\cr}}}
 \smallskip}
\def\tbv#1:#2:#3:#4:#5:{\begingroup\smallskip\parindent0pt
   \centerline{\vbox{\dtmp=\hsize
  \advance\dtmp-2em
  \dtmpi=\dtmp \divide\dtmpi by5 \advance\dtmpi-.5em \def\h[##1]{\hskip.3em\sl##1}
  \setbox1=\vbox{\hsize=\dtmpi#1} \setbox2=\vbox{\hsize=\dtmpi#2}
  \setbox3=\vbox{\hsize=\dtmpi#3}
  \setbox4=\vbox{\hsize=\dtmpi#4} \setbox5=\vbox{\hsize=\dtmpi#5} \dtmpii=0pt
  \ifdim\dtmpii<\ht1 \dtmpii=\ht1\fi \ifdim\dtmpii<\ht2 \dtmpii=\ht2\fi
  \ifdim\dtmpii<\ht3 \dtmpii=\ht3\fi \ifdim\dtmpii<\ht4 \dtmpii=\ht4\fi
  \ifdim\dtmpii<\ht5 \dtmpii=\ht5\fi
  \halign to\dtmp{##\hfil&##\hfil&##\hfil&##\hfil&##\hfil\cr
  \vbox to\dtmpii{\hsize=\dtmpi\unvbox1\vfil} &
  \vbox to\dtmpii{\hsize=\dtmpi\unvbox2\vfil} &
  \vbox to\dtmpii{\hsize=\dtmpi\unvbox3\vfil} &
  \vbox to\dtmpii{\hsize=\dtmpi\unvbox4\vfil} &
  \vbox to\dtmpii{\hsize=\dtmpi\unvbox5\vfil}\cr}}}\endgroup}
\def\tabhead{\def\erule{\vskip.2em\hrule\vskip.2em}
 \def\ptl{\quad} \def\ptr{\hfil\quad}}
\def\trule{\noalign{\vskip.2em\hrule\vskip.2em}}
\def\tbl#1\lbt{\begingroup\dtmp=\hsize \advance\dtmp-3em \smallskip
 \centerline{\vbox{\hsize=\dtmp \halign{###&\quad##\hfil&\quad##\hfil\cr #1}}}
   \endgroup}
\def\fgr[#1](#2)#3\rgf{\begingroup\dtmp=\hsize \divide\dtmp by#1 \advance\dtmp-1em
  \def\pic[##1]{\epsfxsize=\dtmp \epsffile{#2##1.eps}}
 \centerline{\vbox{\halign{###&\hfil##\hfil&##\cr #3\cr}}}\endgroup}
\def\fgl[#1]#2\lgf{\begingroup\dtmp=\hsize \divide\dtmp by#1 \advance\dtmp-1em
  \def\lab[##1]{\hbox to\dtmp{\hfil##1\hfil}}
 \centerline{\vbox{\halign{###&\hfil##\hfil&##\cr #2\cr}}}\endgroup}
% \Pp<lattice>:<Pc type>:<number\(<order>),...>:<Pc prob.>:<std. error>:
\def\Pp#1:#2:#3:#4:#5:{\begingroup \def\(##1){({\bf##1})}
   \def\Vii{Vn 2--d} \def\Viii{Vn 3--d}
 ${}^{\hbox{\sixit#1}}_{\scriptscriptstyle #3}p_#2$ $=$ $#4\pm #5$\endgroup}
\def\X#1:#2:#3:#4:#5:{\begingroup \def\(##1){({\bf##1})}
   \def\Vii{Vn 2--d} \def\Viii{Vn 3--d}
 ${}^{\hbox{\sixit#1}}_{\scriptscriptstyle #3}x_#2$ $=$ $#4\pm #5$\endgroup}
```

```
\def\din#1.#2.#3.#4.{\begingroup
   \def\der[##1]{{\bf ##1},} % #1 \latin, #2 \tenbf , #3 \ninepoint
 \def\w##1:{\dtmpi=\hsize \advance\dtmpi-.3em \dtmpii=\dtmpi \advance\dtmpii-.3em
  \parshape=2 0pt \dtmpi .3em \dtmpii {#2 ##1}}
 \def\Bf[##1]{{\bf##1}} \def\(##1){{\it##1}}
   \def\Sl[##1]{{\sl##1}} \def\[##1]{{\tt##1}}
 #1\dtmp=\hsize
 \begindoublecolumns\hsize=\dtmp \divide\hsize by2 \baselineskip.9em\parindent0pt #3
 \input #4
 \enddoublecolumns\endgroup}
\long\def\t[#1]{\def\next{#1}{\tt\frenchspacing\expandafter\strip\meaning\next}}
 \def\strip#1>{}
\def\idx[#1]{\beginindex\input #1\endindex}
\newbox\partialpage
\def\begindoublecolumns{\dtmp=\hsize \dtmpi=\vsize\begingroup
  \output={\global\setbox\partialpage=\vbox{\unvbox255\bigskip}}\eject
  \output={\doublecolumnout} \hsize=\dtmp \divide\hsize by2 \advance\hsize-.4em
  \vsize=\dtmpi \multiply\vsize by2 \advance\vsize by.4em}
\def\enddoublecolumns{\output={\balancecolumns}\eject
  \endgroup \pagegoal=\vsize}
\def\doublecolumnout{\splittopskip=\topskip \splitmaxdepth=\maxdepth
  \dtmp=\dtmpi \dimen@=\dtmp \advance\dimen@ by-\ht\partialpage
  \setbox0=\vsplit255 to\dimen@ \setbox2=\vsplit255 to\dimen@
  \onepageout\pagesofar
  \unvbox255 \penalty\outputpenalty}
\def\pagesofar{\unvbox\partialpage
  \wd0=\hsize \wd2=\hsize \hbox to\pagewidth{\box0\hfil\box2}}
\def\balancecolumns{\setbox0=\vbox{\unvbox255} \dimen@=\ht0
  \advance\dimen@ by\topskip \advance\dimen@ by-\baselineskip
  \divide\dimen@ by2 \splittopskip=\topskip
  {\vbadness=10000 \loop \global\setbox3=\copy0
    \global\setbox1=\vsplit3 to\dimen@
    \ifdim\ht3>\dimen@ \global\advance\dimen@ by1pt \repeat}
  \setbox0=\vbox to\dimen@{\unvbox1}
  \setbox2=\vbox to\dimen@{\unvbox3}
  \pagesofar}
\def\begintriplecolumns{\dtmp=\hsize \dtmpi=\vsize \begingroup
  \output={\global\setbox\partialpage=\vbox{\unvbox255\bigskip}}\eject
  \output={\triplecolumnout} \hsize=\dtmp \divide\hsize by3 \advance\hsize by-.4em
   \vsize=\dtmpi \multiply\vsize by3 \advance\vsize by.4em}
\def\endtriplecolumns{\output={\balancethreecolumns}\eject
  \endgroup \pagegoal=\vsize}
\def\triplecolumnout{\splittopskip=\topskip \splitmaxdepth=\maxdepth
  \dtmpii=\dtmpi \dimen@=\dtmpii \advance\dimen@ by-\ht\partialpage
  \setbox0=\vsplit255 to\dimen@ \setbox2=\vsplit255 to\dimen@
   \setbox4=\vsplit255 to\dimen@
  \onepageout\pagesofar
  \unvbox255 \penalty\outputpenalty}
\def\pagesofar{\unvbox\partialpage
  \wd0=\hsize \wd2=\hsize \wd4=\hsize \hbox to\pagewidth{\box0\hfil\box2\hfil\box4}}
\def\balancethreecolumns{\setbox0=\vbox{\unvbox255} \dimen@=\ht0
  \advance\dimen@ by\topskip \advance\dimen@ by-\baselineskip
  \divide\dimen@ by3 \splittopskip=\topskip
  {\vbadness=10000 \loop \global\setbox5=\copy0
    \global\setbox1=\vsplit5 to\dimen@
    \global\setbox3=\vsplit5 to\dimen@
    \ifdim\ht5>\dimen@ \global\advance\dimen@ by1pt \repeat}
  \setbox0=\vbox to\dimen@{\unvbox1}
  \setbox2=\vbox to\dimen@{\unvbox3}
  \setbox4=\vbox to\dimen@{\unvbox5}
  \pagesofar}
% cf. manmac.tex
\def\onepageout#1{\shipout\vbox{ % here we define one page of output
    \offinterlineskip % butt the boxes together
    \vbox to 3pc{ % this part goes on top of the 44pc pages
      \iftitle % the next is used for title pages
        \global\titlefalse % reset the titlepage switch
        \setcornerrules % for camera alignment
      \else\ifodd\pageno \rightheadline\else\leftheadline\fi\fi
      \vfill} % this completes the \vbox to 3pc
    \vbox to \pageheight{\dtmp=\hsize \advance\dtmp by1pc
      \ifvoid\margin\else % marginal info is present
        \rlap{\kern\dtmp\vbox to\z@{\kern4pt\box\margin \vss}}\fi
      #1 % now insert the main information
      \ifvoid\footins\else % footnote info is present
        \vskip\skip\footins \kern-3pt
        \hrule height\ruleht width\pagewidth \kern-\ruleht \kern3pt
        \unvbox\footins\fi
      \boxmaxdepth=\maxdepth
      } % this completes the \vbox to \pageheight
    }
  \advancepageno}
```

```
\def\pap#1{\begingroup\parindent=1em \ninerm\input #1\endgroup}
\def\lhd[#1]{\leftline{\(\tenrm\bf #1)}}
\def\chd[#1]{\centerline{\(\tenrm #1)}}
\def\qte[#1][#2]{\begingroup\dtmp=\hsize \advance\dtmp-3em
 \setbox1=\vbox{\hsize=\dtmp \parindent=0pt #1\smallskip\rightline{#2}}
 \smallskip\centerline{\copy1}\smallskip\endgroup}
\def\quo[#1]{\begingroup \def\|{\par} \dtmp=\hsize \advance\dtmp-5em
 \setbox1=\vbox{\hsize=\dtmp \parindent=0pt #1}
 \smallskip\centerline{\copy1}\smallskip\endgroup}
\def\bitm[#1]{\smallskip\begingroup \def\itm{\item{#1}}}
\def\eitm{\endgroup\smallskip}
\def\btab[#1]#2\etab{\centerline{\setbox1=\vbox{#2}
 \vbox{\halign{#1\let|=&
 \let*=\cr
 \copy1}}}}
\def\tdim{\dtmp=\hsize \advance\dtmp by-2em \hsize=\dtmp}
\newcount\apc \apc=64
\def\app[#1]#2:#3:{\advance\apc by 1
 \immediate\write\ref{\string\ix[#1](\char\the\apc)}
 \immediate\write\ref{\string\ix[#1n](#2)}
 \immediate\write\ref{\string\ix[#1p](\number\pageno)}
 \ix[#1](\char\the\apc) \ix[#1n](#2) \ix[#1p](\number\pageno)
 \beginchapter {\Title\ Appendix \char\apc} {}{\char\apc}. {\x[#1n]}\par
 \S\ \Bf \x[#1] \x[#1n]:\par \begingroup\input #3 \endgroup\par}
\newcount\cch
\def\chp[#1]#2:#3:{\advance\cch by 1
 \immediate\write\ref{\string\ix[#1](\the\cch)}
 \immediate\write\ref{\string\ix[#1n](#2)}
 \immediate\write\ref{\string\ix[#1p](\number\pageno)}
 \ix[#1](\the\cch) \ix[#1n](#2) \ix[#1p](\number\pageno)
 \stn=0 \alc=0 \fgc=0 \tbc=0 \cac=0 \tmc=0
 \beginchapter {\Title\ Chapter} {}{\x[#1]}. {\x[#1n]}\par
 \S\ \Bf \x[#1] \x[#1n]:\par \begingroup\input #3 \endgroup\par}
\newcount\stn
\def\sct[#1]#2:#3:{\advance\stn by 1
 \immediate\write\ref{\string\ix[#1](\the\cch.\the\stn)}
 \immediate\write\ref{\string\ix[#1n](#2)}
 \immediate\write\ref{\string\ix[#1p](\number\pageno)}
 \ix[#1](\the\cch.\the\stn) \ix[#1n](#2) \ix[#1p](\number\pageno)
 \beginsection
 \S\ {\bf\x[#1] \x[#1n]}\par \begingroup\input #3 \endgroup\par}
\newcount\alc
\def\alg[#1]#2:{\advance\alc by 1
 \immediate\write\ref{\string\ix[#1](\the\cch.\the\alc)}
 \immediate\write\ref{\string\ix[#1n](#2)}
 \immediate\write\ref{\string\ix[#1p](\number\pageno)}
 \ix[#1](\the\cch.\the\alc) \ix[#1n](#2) \ix[#1p](\number\pageno)}
\newcount\eqc
\def\eqn{\global\advance\eqc by 1
   \eqno{(\bf\the\eqc)_{\hbox{\sevenrm\romannumeral\the\cch}}}}
\def\eql{\global\advance\eqc by 1
   (\bf\the\eqc)_{\hbox{\sevenrm\romannumeral\the\cch}}}
\def\eqa[#1]#2:{\global\advance\eqc by 1
 \immediate\write\ref{\string\ix[#1](\the\cch.\the\eqc)}
 \immediate\write\ref{\string\ix[#1n](#2)}
 \immediate\write\ref{\string\ix[#1p](\number\pageno)}
 \ix[#1](\the\cch.\the\eqc) \ix[#1n](#2) \ix[#1p](\number\pageno)
  \eqno{(\bf\the\eqc)_{\hbox{\sevenrm\romannumeral\the\cch}}}}
\def\eqal[#1]#2:{\global\advance\eqc by 1
 \immediate\write\ref{\string\ix[#1](\the\cch.\the\eqc)}
 \immediate\write\ref{\string\ix[#1n](#2)}
 \immediate\write\ref{\string\ix[#1p](\number\pageno)}
 \ix[#1](\the\cch.\the\eqc) \ix[#1n](#2) \ix[#1p](\number\pageno)
  (\bf\the\eqc)_{\hbox{\sevenrm\romannumeral\the\cch}}}
\newcount\fgc
\def\fig[#1]#2:{\advance\fgc by 1
 \immediate\write\ref{\string\ix[#1](\the\cch.\the\fgc)}
 \immediate\write\ref{\string\ix[#1n](#2)}
 \immediate\write\ref{\string\ix[#1p](\number\pageno)}
 \ix[#1](\the\cch.\the\fgc) \ix[#1n](#2) \ix[#1p](\number\pageno)}
\newcount\tbc
\def\tab[#1]#2:{\advance\tbc by 1
 \immediate\write\ref{\string\ix[#1](\the\cch.\the\tbc)}
 \immediate\write\ref{\string\ix[#1n](#2)}
 \immediate\write\ref{\string\ix[#1p](\number\pageno)}
 \ix[#1](\the\cch.\the\tbc) \ix[#1n](#2) \ix[#1p](\number\pageno)}
\newcount\cac
\def\proclaim #1. #2\par{\medbreak
  \noindent{\bf#1.\enspace}{\sl#2\par}%
  \ifdim\lastskip<\medskipamount \removelastskip\penalty55\medskip\fi}
\def\cly[#1]#2:#3\par{\advance\cac  by 1
 \immediate\write\ref{\string\ix[#1](\the\cch.\the\cac)}
```

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 \immediate\write\ref{\string\ix[#1n](#2)}
 \immediate\write\ref{\string\ix[#1p](\number\pageno)}
 \ix[#1](\the\cch.\the\cac) \ix[#1n](#2) \ix[#1p](\number\pageno)
 \proclaim Corollary \x[#1]. #3\par}
\newcount\tmc
\def\thm[#1]#2:#3\par{\advance\tmc  by 1
 \immediate\write\ref{\string\ix[#1](\the\cch.\the\tmc)}
 \immediate\write\ref{\string\ix[#1n](#2)}
 \immediate\write\ref{\string\ix[#1p](\number\pageno)}
 \ix[#1](\the\cch.\the\tmc) \ix[#1n](#2) \ix[#1p](\number\pageno)
 \proclaim Theorem \x[#1]. #3\par}
\def\xp[#1]{\immediate\write\ref{\string\ix[#1](\number\pageno)}
  \ix[#1](\number\pageno)}
\def\mctb[#1:#2:#3:#4:#5:#6:#7]{\smallskip\begingroup\dtmp=\hsize \divide\dtmp by7
 \advance\dtmp.3em \dtmpi=0pt \setbox1=\vbox{\hsize=\dtmp #2}
 \dtmpii=\ht1 \ifdim\dtmpii>\dtmpi \dtmpi=\dtmpii\fi
 \setbox1=\vbox{\hsize=\dtmp #3} \dtmpii=\ht1
 \ifdim\dtmpii>\dtmpi \dtmpi=\dtmpii\fi \setbox1=\vbox{\hsize=\dtmp #4}
 \dtmpii=\ht1 \ifdim\dtmpii>\dtmpi \dtmpi=\dtmpii\fi
 \setbox1=\vbox{\hsize=\dtmp #5} \dtmpii=\ht1
 \ifdim\dtmpii>\dtmpi \dtmpi=\dtmpii\fi \setbox1=\vbox{\hsize=\dtmp #6}
 \dtmpii=\ht1 \ifdim\dtmpii>\dtmpi \dtmpi=\dtmpii\fi
 \setbox1=\vbox{\hsize=\dtmp #7} \dtmpii=\ht1
 \ifdim\dtmpii>\dtmpi \dtmpi=\dtmpii\fi
 \centerline{\vbox{\halign{##\hfil&\quad##\hfil&##\hfil&##\hfil&
    ##\hfil&##\hfil&##\hfil\cr
  &~\(Cell) &~\(Bond) &~\(cell) &~\(bond) &~\(vertex) &~\(edge) \cr
   #1\cr &&&&&&\cr
  &\vbox to\dtmpi{\hsize=\dtmp\parindent=0pt #2\hfil\vfill}
  &\vbox to\dtmpi{\hsize=\dtmp\parindent=0pt #3\hfil\vfill}
  &\vbox to\dtmpi{\hsize=\dtmp\parindent=0pt #4\hfil\vfill}
  &\vbox to\dtmpi{\hsize=\dtmp\parindent=0pt #5\hfil\vfill}
  &\vbox to\dtmpi{\hsize=\dtmp\parindent=0pt #6\hfil\vfill}
  &\vbox to\dtmpi{\hsize=\dtmp\parindent=0pt #7\hfil\vfill}\cr}}}\endgroup}
\def\mvmm(#1;#2;#3,#4;#5,#6){\begingroup\sevenrm\setbox1=\hbox{#1$\pm$#2}
 \ifx0#4\def\tmpa{#3}\else\def\tmpa{#3\times 10^{#4}}\fi
 \ifx0#6\def\tmpb{#5}\else\def\tmpb{#5\times 10^{#6}}\fi
 $\mathrel{\mathop{\copy1}\limits_{\tmpa\atop\tmpb}}$\endgroup}
\def\cn(#1,#2/#3){\begingroup\sevenrm
\setbox1=\hbox{#1[#2]} \dtmp=\wd1
 \setbox2=\hbox{\fiverm#3} \ifdim\wd2>\dtmp \dtmp=\wd2\fi \setbox2=\hbox{\fiverm#3}
 $\mathop{\copy1}\limits_{\copy2}$\endgroup}
% 2-homohedral tilings
\def\hm#1.{\begingroup\def\(##1){{\rm ##1}} \ifcase#1\or$3_3[3^3]7_1[3^7]$
 \or$3_3[3^3]9_3[3^9]_\(II)$\or$4_4[3^4]8_4[3^8]$\or$3_3[3^3]8_2[3^8]$
 \or$3_3[3^3]9_3[3^9]_\(III)$\or$4_3[3^4]10_6[3^{10}]_\(I)$
 \or$3_3[3^3]9_3[3^9]_\(I)$\or$4_2[3^4]10_4[3^{10}]$
 \or$4_3[3^4]10_6[3^{10}]_\(II)$\or$4_2[3^4]8_2[3^8]$
 \or$4_3[3^4]8_3[3^8]_\(I)$\or$4_3[3^4]8_3[3^8]_\(II)$
 \or$4_3[3^4]8_3[3^8]_\(III)$\or$4_4[3^4]7_2[3^7]_\(II)$
 \or$3_3[3^3]12_6[3^{12}]$\or$4_4[3^4]7_2[3^7]_\(I)$
 \or$5_4[3^5]7_4[3^7]_\(I)$ \or$3_1[4^3]5_1[4^5]_\(I)$
 \or$3_1[4^3]5_1[4^5]_\(II)$\or$5_3[3^5]8_6[3^8]_\(II)$
 \or$5_3[3^5]8_6[3^8]_\(III)$\or$5_2[3^5]12_{12}[3^{12}]$
 \or$5_4[3^5]7_4[3^7]_\(II)$ \or$5_3[3^5]8_6[3^8]_\(I)$
 \or$4_2[3^4]12_6[3^{12}]$\or$4_2[3^4]18_{12}[3^{18}]$
 \or$5_3[3^5]7_3[3^7]_\(I)$ \or$5_3[3^5]7_3[3^7]_\(II)$
 \or$3_2[4^3]5_2[4^5]_\(I)$\or$3_2[4^3]5_2[4^5]_\(II)$
 \or$3_3[4^3]5_3[4^5]$\or$3_1[4^3]6_2[4^6]_\(I)$
 \or$3_1[4^3]6_2[4^6]_\(II)$\or$3_2[4^3]6_4[4^6]$
 \or$3_3[4^3]6_6[4^6]$ \or$3_1[4^3]8_4[4^8]$\or$3_1[5^3]4_2[5^4]_\(I)$
 \or$3_1[5^3]4_2[5^4]$\or$3_2[5^3]4_4[5^4]$\fi\endgroup}
\def\kittix{$\raise.1em\hbox{$>$}$\kern-.2em\raise.118em\hbox{:}Kittix}
\def\blr{\begingroup \dtmp=\hsize \advance\dtmp-1em \dtmpi=\dtmp
 \advance\dtmpi-1em \divide\dtmpi by2 \parindent0pt \def\stn{\smallskip}
 \def\b[##1]{\bf ##1} \def\[##1]{\it ##1}
 \def\l##1:##2:{\setbox0=\vbox{\hsize=\dtmpi\sl##1} \setbox1=\vbox{\hsize=\dtmpi##2}
 \ifnum\ht0>\ht1 \dtmpii=\ht0 \else \dtmpii=\ht1 \fi
 \centerline{\vbox to\dtmpii{\hsize=\dtmpi\sl##1}\hfil
  \vbox to\dtmpii{\hsize=\dtmpi##2}}}}
\def\elr{\endgroup}
\def\wra[#1][#2][#3]{\begingroup\baselineskip=2em
  \setbox1=\hbox{#3}$\aleph(\hbox{#1})=$
  $\bigl\{$[#2], (\copy1)$\bigr\}$\endgroup}
\def\wrs[#1][#2][#3][#4]{$\aleph(\hbox{#1})=$ $\bigl\{$[#2], (#3),
  $\langle$#4$\rangle$ $\bigr\}$}
\input language
\asl\bengali\chem\czech\daiy\deutsch\espanlol\francjais\gaelic\grammar
 \hindi\hungarian\lanna\latin\lating\math\nihongo\norge\pali\physics\polish\russian
 \sanskrit\serbo\slovak\vietnamese\zhongwen
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Bibliography

§ **B. Bibliography**

George O. Abell. The distribution of rich clusters of galaxies. *The Astrophysical Journal. Supplement Series.* **3**, Supplement Number 31, 211–288. May, 1958.

Joan Adler and Amnon Aharony. Diffusion percolation. I. Infinite time limit and bootstrap percolation. *Journal of Physics A. Mathematical and General.* **21**, 1387–1404. 1988.

P. K Ahnelt., E. Fernandez, O. Martinez, J. A. Bolea, A. Kubber-Heiss. Irregular S-cone mosaics in felid retinas. Spatial interaction with axonless horizontal cells, revealed by cross correlation. *Journal of the Optical Society of America. A, Optics, Image Science, and Vision.* **17**, 3, Mar. 2000, p. 580–8.

John Algeo. *Problems in the origins and development of the English language.* 3rd ed. Harcourt Brace Jovanovich. 1966 (1982).

J. Ammermuller, W. Mockel, P. Rujan. A geometrical description of horizontal cell networks in the turtle retina. *Brain Research.* **616**, 1–2, 9^{th} Jul. 1993, p. 351–6.

Alizabeth Andrews. *Healthy practice for musicians.* Rhinegold. London. 1997.

Francis Bacon. *Novum Organum.* (The new organon. or true directions concerning the interpretation of nature.) 1620. Translation. James Spedding, et al. Taggard and Thompson. 1863.

Claude Berge. *The theory of graphs and its applications.* 1958. Translation. Alison Doig. Methuen. Wiley. 1962(1964).

Gavin Betts. *Latin.* Teach Yourself Books. 2000.

Citr Bhumiśakdżi. *The Lahu or Mụzọeŕ language.* [Bąsą Lahu rhụe Mụzọeŕ] Maif.ngąm. Thailand. 1963. (in Thai.)

G. Birkhoff and S. M. Lane. *A survey of modern algebra.* A K Peters. 1997.

Reginald Horace Blyth. *Oriental Humour.* Hokuseido. Tokyo. 1959.

Cao Bohan. The Chinese language movement since the May Fourth Period. *in Language reform in China.* Peter J. Seybolt and Gregory Kuei-ke Chiang. Eds. M. E. Sharp Inc. 1978 (1979).

A. Bravais. *A. Bravais' Abhandlungen über symmetrische polyeder.* (1849). Ostwald's Klassiker der Exakten Wissenschaften. no. 17, 46–49. Wilhelm Engelmann. Leipzig. 1890.

S. R. Broadbent and J. M. Hammersley. Percolation processes I, Crystals and mazes. *Proceedings of the Cambridge Philosophical Society.* 54, 629–641. 1957.

Ronald S. Burt. *Structural holes. The social structures of competition.* Harvard University Press. 1992.

A. G. W. Cameron. Abundance of the elements in the solar system. *Space Science Reviews.* 15, 121–147. 1973.

J. Chalupa, P. L. Leath and G. R. Reich. Bootstrap percolation on a Bethe lattice. *Journal of Physics C, Solid State Physics.* **12**, L31–L35. 1979.

W. D. Clayton. Studies in the *Gramineae.* XXI. *Coelorhachis* and *Rhytachne.* A study in numerical taxonomy. *Kew Bulletin.* **24**, 309–314. 1970.

W. D. Clayton. Studies in the *Gramineae.* XXVI. Numerical taxonomy of the *Arundinelleae. Kew Bulletin.* **26**, 111–123. 1972.

Collins USA *Collins Road Atlas USA. Canada and Mexico.* Rand McNally. 1999.

Ken Croswell. *The alchemy of heaven.* Oxford. 1995.

H. M. Cundy and A. P. Rollett. *Mathematical models.* 3^{rd} edition, 1989. Tarquin.

C. A. Curcio, K. R. Sloan. Packing geometry of human cone photoreceptors, variation with eccentricity and evidence for local anisotropy. *Visual Neuroscience.* Vol. 9, No. 2, Aug. 1992, p. 169–80.

James Dwight Dana and Edward Salisbury Dana. *Dana's new mineralogy.* 8th ed. Wiley. 1997.

P. G. de Gennes, P. Lafore and J. P. Millot. Amas accidentels dans les solutions solides désordonnées. *Journal of Physics and Chemistry of Solids.* **11**, 105–110. 1959.

Renati Des-cartes *Principia philosophiæ.* 1644. Ludovicum Elzevirium. *cf* René Descartes. Œuvres philosophiques. Vol.III. 1973. Garnier Frères. Ferdinard Alquié ed.

G. de Vancouleurs. The extragalactic distance scale VIII. A comparison of distance scales. *The Astrophysical Journal.* **415**, 10–32. 20^{th} September 1993 .

Charles Dickens. *A child's history of England.* 1851(1868).

Arthur Eugene Fitzgerald, Charles Kingsley Jr. and Alexander Kusko. *Electric Machinery.* McGraw-Hill Kogakusha. 3rd Ed. 1971.

James P. Evans and Frederick M. Chester. Fluid-rock interaction in faults of the San Andreas system. In ferences from San Gabriel fault rock geochemistry and micro structures. *Journal of Geophysical Research.* **100**, B7, 13007–13020. 10 July 1995.

Rachel Farmer. *Beginner's Russian. An easy introduction.* Teach Yourself Books. 1996.

Mason Florence, Marisa Gierlich and Andrew Dean Nystrom. *Rocky Mountains. Colorado, Wyoming, Montana and Idaho.* Lonely Planet. 3rd ed. 2001.

E. M. Forster. *Howards End.* 1910.

A. S. Glassner. Ed. *Graphics gems.* Academic Press. 1990.

Nigel Gotteri and Joanna Michalak-Gray. *Polish.* Teach Yourself Books. 1997.

G. Grimmett. *Percolation.* Second edition. Springer. 1999.

Branko Grünbaum and G. C. Shepherd. *Tilings and patterns.* W. H. Freeman. N.Y. 1987.

J. M. Hammersley and W. Morton. Poor man's Monte Carlo. *Journal of the Royal Statistical Society (B).* 16, 23–38. 1954.

Allen Hammond. *Which World? scenarios for the* 21^{st} *century.* Earthscan. 1998.
C. Hammond. *The basics of crystallography and diffraction.* Oxford. 2001.
Thomas L. Hankins. Triplets and triads. Sir William Rowan Hamilton on the metaphysics of mathematics. *Isis.* **68**, 242, 175–193. 1977.
John Happel. Viscous flow relative to arrays of cylinders. *American Institute of Chemical Engineers Journal.* vol. 5, no. 2, 174–177. June, 1959.
H. W. Harrison and P. M. Trotman. *Foundations, basements and external works. Performance, diagnosis, maintenance, repair and the avoidance of defects.* BRE. 2002.
H. S. Hele-Shaw. The flow of water. *Nature.* Vol. 58, May 12, 1898, p. 34–36.
H. S. Hele-Shaw. The motion of a perfect liquid. *Nature.* Vol. 60, September 7, 1899, p. 446–451.
Fred Hoyle. *Ten faces of the universe.* Heinemann. London. 1977.
Fred Hoyle and Chandra Wickramasinghe. *Life cloud* the origin of life in the universe.
J. M. Dent & Sons. 1978.: **Fred Hoyle and Nalin Chandra Wickramasinghe.** *Diseases from space.* J. M. Dent & Sons. 1979.
Charlie Ireland. Private communication. from Invercargill, New Zealand. 12^{th} August 2002
C. Isenberg. *The science of soap films and soap bubbles.* Tieto. 1978.
Harold James. *The German slump. Politics and economics 1924–1936.* Clarendon. Oxford. 1986(1987).
Vera Javarek and Miroslava Sudjić *Serbo-Croat.* 2^{nd} ed. Teach Yourself Books. 1963(1972).
G. R. Jerauld, J. C. Hatfield, L. E. Scriven and H. T. Davis. Percolation and conduction on Voronoi and triangular networks. A case study in topological disorder. *Journal of Physics C. Solid State Physics.* **17**, 1519–1529. 1984.
Immanuel Kant. Beantwortung der Frage. Was ist Aufklärung. *Berlinische Monatsschrift.* **4**. December. 1784. 481–494.
Alan R. Kerstein. Equivalence of the void percolation problem for overlapping spheres and a network problem. *Journal of Physics A. Mathematical and General.* **16**, 3071–3075. 1983.
Daniel Keyes. *The minds of Billy Milligan.* Bantam. 1999.
Hassan K. Khalil. *Nonlinear Systems.* Prentice Hall. 2^{nd} ed. 1996.
Amnad Khitapanna. (Aṃnạc Gịtảbarṇạ) personal communication. 2000.
C. Klein and C. S. Hurlbut, Jr. *Manual of mineralogy.* 21^{st} Edition. Wiley. 1993.
D. E. Knuth. *The TEX.* Addison Wesley. 1970.
J. M. Kosterlitz and D. J. Thouless. Ordering, metastability and phase transitions in two-dimensional systems. *Journal of Physics C. Solid State Physics.* vol. 6, 1181–1203.
Roderic Lakes. Foam structures with a negative Poisson's ratio. *Science.* **235**, 1038–1040. 1987.
A. Levy, S. Reich and P. Meakin. The shape of clusters on rectangular 2D lattices in a simple "phase separation" computer experiment. *Physics Letters.* vol. 87A, no. 5, 248–252. 11^{th} January 1982 .
Larry S. Liebovitch and Daniela Scheurle. Two lessons from fractals to chaos. *Complexity.* vol. 5, no. 4, 34–43. March-April, 2000.
Ian S. Lustick and Dan Miodownik. Deliberative democracy and public discourse. The agent-based argument repertoire model. *Complexity.* vol. 5, no. 4, 13–30. March-April, 2000.
T. Magay and L. Országh. *A concise Hungarian–English dictionary.* Oxford University Press. Akademiai Kiadó 1981 (1990).
K. R. Mecke and A. Seyfried. Strong dependence of percolation thresholds on polydispersity. *Europhysics Letters.* **58** (1), 28–34, 1^{st} April 2002 .
Hans Meinhardt. *The algorithmic beauty of sea shells.* Springer-Verlag. 1995 (1998).
Stephen A. Miler and Amos Nur. Permeability as a toggle switch in fluid-controlled crustal processes. *Earth and Planetary Science Letters.* **183**, 133–146. 2000.
R. E. Miles. A random division of space. *Supplementary Advanced Applied Probability.* V. 243, No. 1. 1972.
Gian Maria Milesi-Ferretti and Assaf Razin. Current account reversals and currency crises. Empirical regularities. in *Currency Crisis.* Paul Krugman. ed. University of Chicago Press. 2000.
U. Mizutani. *Introduction to the electron theory of metals.* Cambridge. 2001.
J. Møller. Random tessellations in $\Re^d$. *Advances in Applied Probability.* vol. 21, p. 37–73, 1989.
M. Mulder. *Basic principles of membrane technology.* Kluwer. 1992.
Andrew Nathaniel Nelson. *Japanese–English Character Dictionary.* Charles E. Tuttle. 1962.
James Naughton. *Colloquial Slovak.* 1997.
Johann Nittmann, Gérard Daccord and H. Eugene Stanley. Fractal growth of viscous fingers, quantitative characterization of a fluid instability phenomenon. *Nature* Vol. 314, 14 March 1985, p. 141–144
A. Okabe, B. Boots and K. Sugihara. *Spatial tessellations concepts and applications of Voronoi diagrams.* Wiley. 1992.
I. M. Oshanina. *Chinese–Russian Dictionary.* [Kitaĭsko–Russkiĭ Slovarb] State Publisher. Foreign and National Lexicographer. [Gosudarstvennoe Izdatelbstvo. Inostrannyĭ i Nacionalbnyĭ Slovareĭ] Moscow. 1955.
Roger Penrose. *The emperor's new mind. Concerning computers, minds, and the law of physics.* 1989.

R. Penrose. Tilings and quasi-crystals. a non-local growth problem? in *Introduction to the mathematics of quasicrystals.* Marko V. Jarić Ed. 53–79. Aperiodicity and order. Vol. 2. Academic Press. 1989.

Petersburg *Investment projects of St. Petersburg* Information and Consulting Centre. International Finance and Enterprise Week. Manchester International Conference Centre. 31st October–2nd November 2001 .

F. C. Phillips. *An introduction to crystallography.* Longmans. 3rd ed. 1949 (1963).

Plato. [**Aristocles**] *Phaedo.* [Phaidon] 360 BC.

Michael de Podesta. *Understanding the properties of matter.* Taylor & Francis. 2nd ed. 2002.

J. G. Powles and N. Quirke. Fractal geometry and Brownian motion. A new parameter to describe molecular motion. *Physical Review Letters.* Vol. 52, No. 18, 1571–1574. 30 April 1984.

Przemyslaw Prusinkiewicz and Aristid Lindenmayer. *The algorithmic beauty of plants.* Springer-Verlag. 1990.

L. R. Pujara and Naresh Shanbhag. Some stability theorems for polygons of polynomials. *IEEE Transactions on Automatic Control.* Vol. 37. No. 11. Nov. 1992. 1845–1849

L. Rai. Pujara. On the pseudoboundary of unstable polytopes of polynomials. *IEEE Transactions on Automatic Control.* **41**, 8. 1188–1190. August 1996.

Boyd Robertson and Iain Taylor. *Gaelic. A complete course for beginners.* Teach Yourself Books. 1993.

Steven Rodelet and Jeffrey Sacs. The onset of the East Asian Financial Crisis. in *Currency Crisis.* Paul Krugman. ed. University of Chicago Press. 2000.

Joan Russell. *Swahili.* Teach Yourself Books. 1996.

Oliver Sacks. *Awakenings.* Piccador. Pan Books. 1973(1982).

Oliver Sacks. *seeing voices. A journey into the world of the deaf.* Piccador. Pan Books. 1989(1990).

George Salmon. *A treatise on the analytic geometry of three dimensions.* revised by Reginald a. P. Rogers. 5th ed. Longman, Green and Co. 1912.

Ralf Schumacher. *A stochastic model to simulate the structure and performance of cellular polymeric membranes in dead end filtration.* Dipolmarbeit. [Ph.D. thesis] Institute für chemische Verfahrenstechnik der Technischen Universität Clausthal and UMIST. 1996.

Madan G. Singh. Ed. *Systems & Control Encyclopedia.* Pergamon. 1987.

John Maynard Smith and Eörs Szathmáry. *The major transitions in evolution.* W. H. Freeman. 1995.

Herbert Warington Smyth. *Five years in Siam, from 1891–1896.* J. Murray. London. 1898. in two volumes.

Rupert Snell. *Beginner's Hindi script.* Teach Yourself Books. 2000.

Murray R. Spiegel. *Probability and Statistics.* Schaum's outline. McGraw-Hill. 1975.

Dietrich Stauffer and Amnon Aharony. *Introduction to percolation theory.* 2nd ed. Taylor & Francis. 1992. and 2nd revised ed. 1994(1998)

R. L. Stevenson. *Dr. Jenkyll and Mr. Hyde.* 1886.

Robert Louis Stevenson. *The amateur emigrant.* 1895.

William C. Stokoe Jr., Dorothy C. Casterline and Carl G. Croneberg. *A Dictionary of American Sign Language on Liguistic Principles.* Gallaudet College Press. 1965

Aspasia Theodosiou. personal communication. 2002.

J. R. R. Tolkien. *Lord of the Rings.* 1955.

U. of Man. *Mathematics, the University of Manchester, 2000 and 2001 entry.* booklet. the University of Manchester. 2000.

R. van de Weygaert and V. Icke. Fragmenting the universe II. Voronoi vertices as Abell clusters. *Astronomy and Astrophysics.* **213** 1–9. 1989.

Tuan Duc Vuong and John Moore. *Colloquial Vietnamese. A complete language course.* Routledge. 1994.

Arnett Wilkes and Nicholias Nkosi. *Zulu. A complete course for beginners.* Teach Yourself Books. 1995.

Rüdiger Vaas. Ed. Urknall für Einsteiger. *Bild der Wissenschaft.* May 2002.

G. F. Voronoi*i.e.*. New application of continuous parameters to the theory of quadratic form. First memoir. On some properties of the perfect positive quadratic forms. *Journal für die reine und angewandte Mathematik.* **133**, 97–178, 1908. *cf* Tiyapan (2001).

G. F. Voronoi*i.e.*. New application of continuous parameters to the theory of quadratic form. Second memoir. Research on the primitive parallelohedron. *Journal für die reine und angewandte Mathematik.* **134**, 1908. *cf* Tiyapan (2001).

G. F. Voronoi*i.e.*. New application of continuous parameters to the theory of quadratic form. Second memoir. Studies on the primitive parallelohedra. *Journal für die reine und angewandte Mathematik.* **136**, 67–181, 1909. *cf* . Tiyapan (2001).

N. Walsh. *Making TEX work.* O'Reilly. 1994.

David Ward. Language cull could leave people speechless. *The Guardian.* Saturday 25th May 2002 . National news, 13.

Magnus Joseph Wenninger. *Polyhedron models.* Cambridge University Press. 1971.

Angela Wilkes and John Shackell. *Welsh for beginners.* 1989.

David Wilkinson and Jorge F. Willemsen. Invasion percolation. a new form of percolation theory. *Journal of Physics A. Mathematical and General.* **16**, 3365–3376. 1983.
Trevor Williams and Rolf Bjerknes. Stochastic model for abnormal clone spread through epithelial basal layer. *Nature.* vol. 236, 19–21. 3^{rd} March 1972 .
X. J. Zhan, J. B. Troy. Modeling cat retinal beta-cell arrays. *Visual Neuroscience.* Vol. 17, No. 1, Jan.–Feb. 2000, p. 23–39.

Translation

§ C. Translation

§ 7.9 A. Bravais, 1849

[13] Notice
on
the symmetrical polyhedra of geometry
by
A. Bravais
Professor at the polytechnic school

One calls two polyhedra symmetrica, if they are constructed in the same fashion, one above, the other below a plane, with the condition that their homologous vertices are equally farther apart from this plane and lie on the one plane normal to this plane (*Legendre*, Géometrie bk. 6).

I call inverse polyhedra two polyhedra, whose homologous vertices lie at the same distance from a given point, on the one straight line passing through this point, but on opposite side.

I will indicate the first one, as supposed given polyhedron with P and with p its inverse: One sees, that inverted P will be the inverse of p.

I will call symmetry pole of two polyhedra the point, through which all straight lines pass, which pairwisely connect the homologous vertices of the two polyhedra.

If one imagine from the two polyhedra P, p a single polyhedron (P,p) built, then one calls the said point: symmetry centre of teh polyhedron.

Statement I. *If the symmetry pole of teh considered-fixed polyhedron P moves, then the inverse p changes into a new inverse p', and p is always led by a simple translation general in all vertices into p'.*

Let C, Fig. 1, be the first, C' the second symmetry pole, [14] and S an arbitrary vertex of the fixed polyhedron P. Let s be its homologous point of the shifting of C, and s' the same after this shifting.

From $Cs = CS$, $C's' = C'S$ [it] follows, that $ss' = 2CC'$, further, that ss' is parallel with CC'. Let likewisely t and t' be the two locations, which another vertex of the inverse polyhedron takes after each other. One would have then in the same manner $tt' = 2CC'$, tt' parallel with CC'. If one then shifts the polyhedron, as one moves it in the direction from C towards C', precisely parallel with CC', and by an amount $= 2CC'$, then it reaches the agreement with the polyhedron p'. *q.e.d.*

Addendum. It follows from this, that, if a polyhedron P is given, its inverse, both the form of it, and the direction of its parts with regard to the absolute space concerned, is perfectly determined; however the location, that it will take, indeterminate and depends upon the location of the symmetry pole.

Statement II. *In two inverse polyhedra the homologous faces are pairwisely equal to one another, and the inclination of two neighbouring faces in one of these two bodies is equal to the inclination of the homologous faces in the other.*

This Statement can be proved as Statement II of the sixth book by *Legendre*, where one takes vertex-triangle instead of the trapeziums with equal base used by the proof. But one can also proceed in simpler form as follows:

There be the prove, that both the edges, as well as the edge angles and the face angles, which makes the bodily vertex S, Fig. 2, are the same, as their homologous ones in the inverse polyhedron. Here the choice of the symmetry poles remains free, we take S as pole, extend the vertices SM, SN, SP, SQ by $Sm = SM$, $Sn = SN$, $Sp = SP$, $Sq = SQ$ etc. The two opposite bodily vertices will from construction apparently have edges pairwisely the same among one another, their edge angles will be pairwisely the same as vertex angles, and their face angles will also be as large as vertex angle; now both these edges, as well as the edge- and face-angles in the polyhedron and in its inverse are homologous to one another. The same proof can be employed for all bodily vertices, consequently also for all edge- and face-angles. Therefore the faces of two polyhedra are the same and equally inclined against one another. *q.e.d.*

[15] **Statement III.** *If one turns the polyhedron p, the inverse of P, by two right angles around a line passing through a symmetry pole C, then the polyhedron p', which one obtains in this manner, will be symmetrical to P with regard to the plane, which is placed through C normal to the rotation axis.*

Let NN', Fig. 3, be the given straight line and QCQ' the plane placed perpendicular to it through C; let S be an arbitrary vertex of the polyhedron P, and s its homolog in the inverse polyhedron. One drops from s the perpendicular sr on NN' and extend the same to s' by the distance $rs' = rs$. It is obvious, that by a half turning around NN', s will come to s'. I join S with s'. Here $sC = CS$, $sr = rs'$, therefore Ss', will be parallel with NN' and consequently normal to the plane QQ'. If R is the intersection point of Ss' with the plane QQ', then the straight line CR will be normal to NN' and consequently parallel with ss'. Now one has $sC = CS$, therefore, also $SR = Rs'$. Consequently s' is the homologous vertex of S in the polyhedron, that is symmetrically constructed under the plane QQ' with regard to P. The same would be the case with every other vertex of the polyhedron P. Therefore a revolution of two right angles around NN' will bring the inverse p to agreement with p', the symmetrical of P with regard to the plane QQ'. *q.e.d.*

Remark. The plane QCQ' can be called the symmetry plane of the polyhedron (P,p'), if one takes (P,p') as a sole polyhedron.

Statement IV. *Inversely p' will also be the symmetrical of P with regard to an arbitrary plane QQ', if one turns it by two right angles around a normal to the plane, to become one of the inverses of P, and the symmetry pole lies then in the intersection point C of the plane and the rotation axis.*

Because if one constructs the inverse p, as one takes C to the symmetry pole, then p and p' can be brought into agreement, p with p' or p' with p, by a revolution of two right angles around NN'.

Addendum I. It follows from the two previous Statements, that the different polyhedra symmetrical to P, which their form falls upon, are nothing but its inverse polyhedron, and that the same, which the direction

of their parts falls upon, are equivalent to the different polyhedra, which are obtained by revolution of the inverse p by two right angles around freely chosen axes.

[16] Two inverse polyhedra of a given polyhedron P are always identically directed in space. This does not hold with two symmetrical polyhedra of P, taken when the two symmetry planes, which they determine, are parallel to each other.

Addendum II. Two polyhedra, that are symmetrical to P with regard to two freely chosen planes, can always be brought into agreement, because they can both be brought into agreement with an inverse polyhedron P.

Statement V. *In two symmetrical polyhedra the homologous faces are pairwisely the same with one another. (The remaining is as in Statement II.)*

This is the second Proposition of the 6^th book of *Legendre*. It is obvious, here the symmetrical polyhedron can always coincide with the inverse, and since the inverse, in relation to the original polyhedron, from our Statement II possesses the properties described here.

As for the rest the proof of this Statement is unnecessary.

Statement VI. *Two vertex corners are inverse to each other.*

In fact their common vertex is their symmetry pole.

Statement VII. *The plane, which goes through two opposite edges of a parallelepiped, divides it into two inverse prisms, and the bodily vertices lying opposite to one another are inverse to one another.*

One shows firstly, that the diagonals intersect one another in the same point. This point is therefore the symmetry centre of the parallelepiped.

If one then views the two prisms as two different polyhedra, then the symmetry centre will be a symmetry pole, therefore the bodily vertices lying opposite are also inverse, exactly the same as the two prisms. (Statements V and VI of the 6^th book of Géometrie by *Legendre.*)

Statement VIII. *On the sphere every spherical polygon P has its inverse p, the vertices of which lie diametrically opposite to those of the given polygon. If one turns p by 180° around a diameter of the sphere, then [17] the new polygon will be symmetrical to the polygon P with regard to the plane of that greatest circle, which is normal to the diameter chosen as rotary axis.*

This follows from this, that the middle point of the sphere is taken to the symmetry pole.

Inversely every symmetrical polygon of the spherical polygon P can be brought into agreement with its inverse p, via a half turning around the diameter normal to the symmetry plane.

Treatise on the polyhedra of symmetric form
by A. Bravais, Professor at the polytechnicap school. (1849)

The number of the axes of the same kind will be given by the coefficients, which precede the symbolic letters for the axes; therefore the notation $\left[\Lambda^6, 3L^2, 3L'^2\right]$ will mean a six-time major axis, which is connected with three dual axes of a known kind, and three other dual axes of another kind.

The symmetry planes will be indicated by the letter Π, P, P'; we will take the letter Π for the symmetry plane, which is normal to the major axis Λ, the symbols P and P' for symmetry planes, which are perpendicular to no axis of the polyhedron, and the symbols $P^q P'^q P^{q'}$ for the symmetry planes, which lie normal to the axes L^q, L'^q, $L^{q'}$ of this polyhedron. The greatest number of kinds of these planes never exceed three.

The number of the symmetry planes of the same kind is, as with the axes, established by the coefficients which precedes the symbol of these planes. Therefore $\left[\Pi, 3P^{\frac{1}{2}}, 3P'^2\right]$ will then indicate: a symmetry plane, which is normal to the major axis, three symmetry planes of the same kind, normal to the axes $3L^2$, and three symmetry planes of another kind, which are normal to the axes $3L'^2$.

Definition IX and divisions. *With respect to their symmetry the polyhedra have been divided into four large classes:*

1. Asymmetrical polyhedra;

2. Symmetrical polyhedra without axes;

[26] **3. Symmetrical polyhedra with major axis**; this class falls into two divisions: polyhedra with major axis of even order and polyhedra with major axis of odd order.

4. Symmetrical spheroidical polyhedra, which possess one or more axes, of which none is a mojor axis. These are divided into two groups; the quaternary polyhedra and the decemternary polyhedra, depending on the number of their singular ternary axes.

In the studies, which we will make on the polyhedra, we will totally disregard their faces and their edges, in order to look only at their vertices, so that each polyhedron for us will become a set of various points, the number of which is a limited one and which are spread around their centre of gravity in a certain manner.

Definition I. I will call symmetry centre of a polyhedron a point C, Fig. 1, if, when I connect this point with any one vertex S of the polyhedron and extend CS by a magnitude equals to itself, the point s thus obtained is also a vertex of the polyhedron. This point s will be the homolog of S with regard to the middle point C.

Statement I. *In every bounded polyhedron there can arise only one symmetry centre.*

The truth of this assertion is evident.

Definition II. I will call symmetry axis of a polyhedron a straight line AB, Fig. 1, if with a revolution of the polyhedron by an angle Q around AB the new positions of the vertices coincide with the former ones. If for example this rotation brings the vertex S to S', then S' must also be the location of a vertex of the polyhedron, and then S and S' will be called homolog to each other with regard to the axis AB.

[**22**] **Statement II.** *The angle, which by a revolution of the polyhedron around a symmetry axis the locations of the vertices return to themselves, is always commensurate with* 360 *degrees.*

In deed: let S', Fig. 1, be a homolog of S with regard to the axis AB; one places through S and S' a plane normal to AB, which the axis will intersect with c; from c one describes with the radius cS a circle and makes

$$\text{arc}\, S''S' = \text{arc}\, SS', \quad \text{arc}\, S'''S'' = \text{arc}\, SS', \quad etc$$

As the revolution takes the end point S from S to S', the vertex S', which takes part in this movement, will fall on S'', which will also be the location of a vertex. In this manner not only will S and S' be vertices of the polyhedron, bu also S'' and another point S''', ..., which are found in teh same way. As one repeats the arc SS' in the same fashion, one mus come back by one or more circuits to the start point S, otherwise the number of the vertices would be unbounded, which is not possible. Therefore, if one indicates the angle SCS' with K, and with p and q two relative primes, then:

$$K = \frac{p}{q}360^\circ.$$

Addendum. The smallest among the rotation angle, which is capable, to bring the locations of the vertices back to themselves, is $\frac{360^\circ}{q}$. In fact the general expression for this angle certainly is

$$mK - n \cdot 360^\circ = \frac{mp - nq}{q}360^\circ,$$

wherein m and n are integers. Now one can then always fix m and n, that they satisfy the condition

$$mp - nq = \pm 1,$$

consequenty *etc.*

Definition III. One can therefore define the symmetry axis as one such straight line, around which a revolution, in an aliquot part $\frac{1}{q}$ of 360°, the position of the vertices of the polyhedron does not seem to appear different.

[23] The denominator q shall be called the order number of the symmetry of the axis. For $q = 2$, the axis shall be called symmetry axis of the second order or binary†

Statement III. *If two or more symmetry axes exist, then these axes and the symmetry planes, which the polyhedron sometimes has, all intersect one another in the same point.*

Because the centre of gravity of the vertices of the polyhedron, if we take these as equally heavy, must obviously, from the well known construction of the point of gravity, lie on every symmetry axis and also on all symmetry planes of the polyhedron.

Definition VI. The point, in which the axes and planes of symmetry of the polyhedron mutually meet, shall be called centre of the form‡ of the polyhedron. When one sole symmetry axis exists, if all symmetry planes pass through [24] this axis and no symmetry centre exists, then as few exists a centre of the form.

The centre of a regular tetrahedron is a centre of the form, but no symmetry centre for this polyhedron.

Definition VII. Two symmetry axes of the same order shall be called the same kind, if the arrangement of the vertices around the one the same [sic] is as around the other. In order to establish the conformity of this arrangement, one connects in imagination the vertices of the polyhedron with the two axes, and imagine the one of these two systems movable. If then at the same time the movable axis can be brought to the cover with the fixed one and the movable vertices with the fixed ones, then the axes are called of the same kind and directly different.

If the polyhedra, when the axes coincide, can not be brought to agreement, if they are only homologous with reference to a known symmetry plane, ie. if in the meaning taken from *Legendre* (Éléments de Géométrie) one is symmetrical to the other, then the axes will always still be of the same kind, but they shall then be called inversely different.

Two symmetry planes are of the same kind and directly different, if when one turns one of the two around their common intersection line and the polyhedron has taken part, the location of the vertices reach agreement, so long as the first-known symmetry plane coincides with the other. If otherwise the symmetrical conformity of the geometer takes the place of agreement conformity, then the symmetry planes will be called of the same kind, but inversely different.

If these two kinds of difference fail, then one calls theaxes or planes of symmetry such ones of different kind.

Two axes of the same kind are necessarily of the same order; but the opposite is not necessarily the case.

Definition VIII. I call major axis a symmetry axis, to the whole other axes, if it gives still others, [which] are mormal, and to the whole symmetry planes, if it gives those, [which] lie parallel or normal — suppose more over, that the order of symmetry of this major axis be no lower than that of the symmetry of teh other axes.

If two or more axes satisfy these conditions, then one [25] of them can be arbitrarily chosen and looked upon as major axis of the polyhedron.

Notations. In order to symbolically establish the different kinds of symmetry, which the polyhedra can possess, I choose the letter C, to indicate a symmetry centre; $0C$ shall mean, that the polyhedron has no such centre.

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† The description binary [binär], ternary [ternär] *etc* are used in stead of the preferable translation of *Sohncke*'s (dual [zweizählig], tripple [dreizählig] *etc*), in order to be able to accommodate the combinations as quaterternary [quaterternär] and decemtenary [decemternär].

‡ Centre of the form (centre de figure) one has now coined the usual expression *geometrical middlepoint*, in order to preserve as much as possible the contrast with symmetry centre (centre de symétrie).

The letters Λ, L, L' shall indicate symmetry axes; Λ^2, L^2, L'^2 dual axes; Λ^3, L^3 ... triple and so fort, whereby the index on top gives the order number of the symmetry.

The letter Λ shall always refer to the major axis.

These notations suffice for the axes, as no more than three different kinds of them can occur in a polyhedron.

§ I. Asymmetrical polyhedra

Here these polyhedra neither have axes, nor centre, nor planes of symmetry, they can from the preceding definitions be represented by the symbol

$$[0L, 0C, 0P].$$

§ II. Symmetrical polyhedra without axes

Statement IV. *In every polyhedron, that possesses a symmetry plane and a symmetry centre, the straight line, which is placed through this centre normal to the plane, is a symmetry axis of even order.*

Let PQ, Fig. 2, be the symmetry plane, C the centre, S any vertex of the polyhedron, CaA the normal to this plane. Through this straight line and S one places the plane ACS normal to PQ, which contains the vertex s, the homolog of S with regard to the centre C, in exactly the same way as the vertex S', the homolog of s with regard to the plane PQ. If one S and S', then it is certain, that the connection straight-line will be perpendicular to CA and that $aS' = aS$ will hold. The condition for this, that the axis AC be a binary one, is thus fulfilled. The axis AC could again also be a quaternary, senary, in general, an axis of order $2q$. Consequently etc.

Statement V. *If two symmetry planes in a polyhedron exist, then their intersection line is a symmetry axis.*

Let S, Fig. 3, be any vertex of the polyhedron. One takes the plane, which normal to the two given symmetry planes is placed through S, to the plane of the drawing; let CP and Cp be the trace of this plane on the plane of the drawing. One will obtain s, [27] the homolog of S with regard to the plane CP, as one makes in the plane PCp

$$\text{angle } SCs = 2 \cdot \text{angle } SCP = 2 \cdot \text{angle } sCP, \qquad \text{and} \qquad Cs = CS.$$

In the same way one will obtain the homolog S' of s with regard to the plane Cp, if one makes

$$\text{angle } SCs = 2 \cdot \text{angle } sCp \qquad \text{and} \qquad CS' = Cs,$$

from which by substraction follows:

$$S'CS = 2PCp, CS' = CS.$$

When one repeats the construction, which is done with S, with the vertex S', one will likewise come up with another homolog S'', which in exactly the same way is determined by the following equations for the pole:

$$S''CS' = 2PCp, CS'' = CS'.$$

If one then describes from C with a radius CS the circle $SS'S''$ and the arc SS' puts on a known number on this circumference, then the points $SS'S''$ etc, which necessarily make a bounded system, the vertices of a registered one, will be a polygon circumscribed by this circle. If q is the number of these vertices, then one sees, that to each point S, $q-1$ other points, that are homolog to S with regard to the normal to this plane, will correspond. This normal will therefore be a symmetry axis, whose order number q will depend on the value of the angle PCp.

Addendum. The angle SCS' is necessarily of the form

$$\frac{p}{q}360°,$$

wherein p and q are relative prime numbers, and in the case, where S and S' are two homologs as close to each other as possible, one has

$$SCS' = \frac{360°}{q},$$

from the addendum to Proposition II.

Statement VI. *The symmetrical polyhedra without axes have only two different kinds of symmetry, depending on whether they possess a symmetry centre or a symmetry plane.*

[28] Because they can not, from Statement IV, at the same time have a symmetry centre and a symmetry plane, even less so from Proposition V two symmetry planes. The symbols for these two kinds of symmetry will then be:

$$[0L, C, 0P]$$

$$[0L, 0C, P].$$

§ III. Symmetrical polyhedra with major axis

Statement VII. *If a polyhedron has two symmetry planes P and p, that are not normal to each other, then it has a third one P', which is the homolog of P with respect to the plane p and in the same kind as P.*

Let CP and Cp, Fig. 3, be the tracks of the two planes P and p on a plane normal to their mutual intersection line. Let CP' further be the track of the plane P' homologous to P, in relation with the plane Cp lying in between. The vertex s' will have a homolog in S with regard to Cp, and S a homolog s, on the other side of CP. Likewise s will have a homolog in S' of s' with respect to Cp; it is easy to see, that S' will be the homolog of s' with respect to the plane CP'. In this manner has each vertex one other homologous of its own, with regard to this last plane; therefore P' is also a symmetry plane and, as one sees, of the same kind as P.

Statement VIII. *If a polyhedron has a symmetry plane P and a symmetry axis L placed diagonal to it, then the straight line L', the homolog of L with respect to every plane, is also a symmetry axis.*

Because S, Fig. 4, usually is a vertex, which has in s a homolog with regard to the plane P, and s, s', s'', etc. present the system of the homologs of s with regard to the axis L. Let S, S', S'' etc. be the homologs of s, s', s'' etc. with respect to the plane P. The order of S, S', S'' etc. around L' will be the same as that of s, s', s'' etc. around L, therefore L' is also a symmetry axis and of the same kind as L.

Addendum. Every symmetry plane brings about, that not only every vertex, but also every [29] plane or axis of symmetry of the polyhedron repeats itself on the other side of the plane. The planes and axes, which come into being as such, are similarly of the same kind as the original ones and inverse to them.

Statement IX. *In every polyhedron, which has an axis of the order q, correspond to every symmetry plane placed diagonal to the axis $q-1$ other symmetry planes of the same kind.*

This Statement can be proved, as the Proposition VII and VIII.

One can also confine oneself to the notion, that during the revolution of the polyhedron around the axis L^q, the symmetry plane can be viewed as being involved in this revolution; now this plane, while the revolution takes place, does not cease, to be a symmetry plane for the polyhedron, which takes part in its motion. This goes therefore also for the case, that the revolution reaches a sum of $\frac{360^\circ}{q}$.

Statement X. *In every polyhedron, which has an axis L of the order q, correspond to each symmetry axis L' placed diagonal to the first one, $q-1$ other axes of the same order and the same kind as the axis L'.*

This Statement could be proved as the previous one. If the polyhedron is turned by $\frac{360^\circ}{q}$, then the axis does not cease, to be a symmetry axis of the moving polyhedron.

Addendum. Every summetry axis of the order q presupposes the co-existence of all symmetry axes or planes, which are the homologs of a given axis or plane with regard to the axis of the order q. The homologous planes or axes obtained in this manner are of the same kind and directly different among themselves.

Statement XI. *If in total q symmetry planes exist which intersect one another in one and the same straight line, then this straight line is a symmetry axis, whose order number is q or a multiple of q.*

The face angles of these planes are necessarily equal among themselves, because without this these faces would symmetrically repeat with respect to one another (Statement VII), and their number would exceed q.

[30] Let therefore CPA and CpA, Fig. 5, be two neighbouring symmetry planes, then one will evidently have

$$PCp = \frac{180^\circ}{q}.$$

If now the point σ is the homolog of S with regard to the plane PCA, and S' the homolog of σ with regard to the plane pCA, then the revolution, which brings S to S', so that it revolves the polyhedron around CA, will be from Statement V. [sic]

$$PCP' = 2PCp = \frac{360^\circ}{q}.$$

Therefore the axis CA will have the symmetry order q or a multiple of it.

Statement XII. *If the two binary symmetry axes exist, then the normal to their plane placed through their intersection point is a symmetry axis of the polyhedron.*

Let CP and Cp, Fig. 3, be the two binary axes, and S a vertex of the polyhedron. I will assume, that S lies at a height of Δ aobve the plane PCp, the plane of the drawing, and orthogonally projects itself in S; then the homolog of S with regard to the axis CP will be the point s, which lies at a distance Δ under the plane of the drawing, and one will have between the projection S and s the relationship

$$\text{angle}\, SCs = 2\,\text{angle}\, SCP \qquad \text{and} \qquad Cs = CS.$$

In the same way one will obtain the homolog S' of s with regard to the axis Cp, where one sets

$$\text{angle}\, S'Cs = 2\,\text{angle}\, sCP \qquad \text{and} \qquad CS' = Cs,$$

and the point S' will be at a height Δ above the plane of the drawing; therefore

$$S'CS = 2PCp \qquad \text{and} \qquad CS' = CS.$$

When one repeats the same operations with S', one obtains S'', then S'''. All these points will make the vertices of a regular polygon, which is registered to the circle of radius CS, and they will be the homologs of S with regard to the normal [31] to the plane. Therefore this normal will be a symmetry axis, whose order number q depends on the value of the angle PCp.

Statement XIII. *If on a plane is divided into a total number q of binary axes, then the normal to this plane is a symmetry axis, whose order number is q or a multiple of q.*

The angles of these axes are necessarily equal among themselves, otherwise they would oppositely reproduce themselves in the same plane (Statement X, Addendum) and their number would be greater than the number q.

Let therefore CP and Cp be two neighbouring binary axes, Fig. 5, and CA the normal to their plane, then one will evidently have:

$$PCp = \frac{180^\circ}{q}.$$

If now s is the homolog of S with regard to the axis CP, and S' the homolog of s with regard to the axis Cp, then the rotation, which brings S to S', where it turns the polyhedron around CA, from the explanation of the last Statement, will come to:

$$\text{angle } PCP' = 2 \cdot PCp = \frac{360^\circ}{q}.$$

Therefore the straight line CA will be a symmetry axis of the order q or a multiple of q.

Statement XIV. *I three quaternary axes perpendicular to one another exist, then there exist at the same time four ternary axes outside of the planes, which pairwisely bind the quaternary axes.*

Let OA, OB and OC, Fig. 6, be the three quaternary axes, which meet the centre of the form of the polyhedron in O. We let the polyhedron make a quater-turn around OA admittedly from B to C. The apparent position of the vertex will remain the same, and the axis OB will come to OC. We let the polyhedron make a second quarter of a turn, that is around the vertical OC, from A to B. The apparent position of the vertex will still always be the same, the axis OB will remain in OC, and the axis OA will move to OB. The resultant of these double turn will therefore be, that the system of the [32] axes OA and OB fastly bound to the polyhedron and moving with it will be moved on to the system of the fixed staight lines OB and OC. Now it is certain, that this changing of place amounts to a single turn of 120° around the straight line OD, which binds the centre of the form O with the middlepoint D, of the spherical triangle ABC having three right angles. One sees from this, that the line OD is a ternary axis of the polyhedron, and here there are eight triangles with three right angles, whose middlepoint pairwisely lie opposite one another, thus give four such ternary axes, all outside of the planes AOB, AOC and BOC.

Addendum. A polyhedron with three quaternary axes perpendicular to one another has no major axis, because in every polyhedron, which has a major axis, at least two of the three angles, which build three arbitrary axis L, L' and L'', not lying in the same plane, from the definition of the major axis must be equal to 90°. Now the system of the three axes OA, OB and OD does not satisfy this condition.

Statement XV. *In every polyhedron, which has a major axis Λ^q, if a second symmetry axis exists, this axis, which necessarily lies in a plane normal to Λ^q, can only be of binary symmetry.*

Let $C\Lambda$, Fig. 7, be the axis Λ^q, CL the second axis with the unknown order number of its symmetry x. It is clear, that this axis will lie in a plane normal to Λ^q, and I maintain, that $x = 2$ must hold.

In deed from the definition of the major axis the axes homologous to Λ^q with respect to L^x must (Statement X, Addendum) stand perpendicular to Λ^q, or coincide with it. Therefore one can only assume $x = 2$ or $x = 4$, which in the first case results from a half-, in the second case a quarter-turn around L^x.

If $x = 4$, then the axis CL will be quaternary, and the axis $C\Lambda$ will repeat itself in CL', which must be all axis of the same kind as itself (Statement X) and lies in the plane normal to $C\Lambda$ and passing through C. CL' will then also be an axis of the order q, and as a consequence the axis homologous to CL with respect to CL' does not go outside the plane LCL', as the definition of the major axis demands, therefore necessarily $q = 4$ or $q = 2$.

The case $q = 4$ corresponds with the case of the three perpendicular quaternary axis [33] and must be discarded, because it would give no major axis (Statement XIV, Addendum).

In the case $q = 2$ the axis CL will be the genuine major axis. It is in fact easy to see, that it can give no symmetry plane, which goes through $C\Lambda$ and lies oblique to $C\Lambda$, because its first homolog, with regard to the quaternary axis CL, would lie neither normal nor parallel to $C\Lambda$, and that would be inconsistent with the supposition, that $C\Lambda$ is a major axis. Nothing thus stand oppose to the supposition, that we look upon CL as major axis, and this we must do, since its symmetry order is a higher one, than the symmetry order of $C\Lambda$. Consequently, if $C\Lambda$ is really a major axis, it can never be that $x = 4$.

There remains therefore only the supposition $x = 2$; the second symmetry axis will consequently be a simple binary axis.

Statement XVI. *In every polyhedron with major axis the q binary axes, which lie in the plane normal to the major axis, are each equally inclined against the neighbouring ones and are by turns of the same kind.*

Because if CP an Cp, Fig. 3, be two neighbouring binary axes, then the binary axis Cp will force the axis CP, to repeat itself on the opposite side in CP' (Statement X, Addendum). CP' draws for itself the existence of the axis Cp' to itself, and so forth. From this one sees, that the q angles PCp, pCP', $P'Cp'$ etc. must be the same with one another and $= \frac{180^\circ}{q}$.

This axes CP and CP' are of the same kind and directly different from each other with respect to the axis Cp lying between them. Likewise Cp and Cp' are under the same kind of each other and each other directly different with respect to the axis CP', as a consequence etc.

Statement XVII. *The polyhedra, which have an axis Λ^q, further q symmetry planes, which pass through this axis, and the symmetry plane Π normal to the major axis, have at the same time q binary axes in the intersection lines of these q planes with the plane Π.*

The intersection of two symmetry planes is always a symmetry axis (Statement V); now this axis can only be binary (Statement XV), as a consequence etc.

One can, to prove this proposition, also recall Figure 5, where CPQ is one of the q planes, which pass through the axis Λ^q and CPP' the plane Π.

[34] Let Σ be the homolog of the vertex S with regard to the plane Π, let s be the homolog of Σ with regard to the plane CPQ. It is obvious, that S and s are homolog to each other with respect to CP, which therefore is a binary axis of the polyhedron.

Statement XVIII. *The polyhedron, which have an axis Λ^q, further q binary axes normal to Λ^q, and the symmetry plane Π normal to Λ^q, have also q symmetry planes, which pass through the axis Λ^q and through each of the binary axes.*

Let CP, Fig. 5, be the binary axis, let CPP' be the plane Π normal to the major axis CA. The vertex S will have a homolog in s on the other side of the binary axis CP; the vertex s will have a homolog in σ on the other side of the plane Π: the point S and σ will be homologous to each other with respact to the plane CPQ; consequently will finally be one of the symmetry planes of the polyhedron.

Statement XIX. *The polyhedron, which have a major axis Λ^q, further q symmetry axes, which pass through this axis, and a symmetry centre, have at the same time q binary axes, which lie through the centre normal to one of these planes.*

This is a consequence of the Statements IV and XV.

Statement XX. *The polyhedra, which have an axis Λ^q, further q binary axes, normal to Λ^q, and a symmetry centre, have also q symmetry planes, which are normal to these binary axes.*

This follows from Statement XXI, whose proof follows.

Polyhedra with major axis of even order

Statement XXI. *Every polyhedron, which has an axis of even order and a symmetry centre, has a symmetry plane, which passes through the centre and lies normal to each axis.*

Let CA, Fig. 2, be the symmetry axis of the order $2q$, C the symmetry centre and PQ the plane normal to CA. The vertex S has $2q-1$ homologs with respect to the axis CA; if one thus drops the normal Sa on to CA and extends it by $aS' = aS$, then S' will be one of these homologs. As one connects S' with the centre C and extend by $c\sigma = cS'$, one obtains the vertex Σ, the homolog of S' with respect to the middlepoint C. Apparently S and Σ are homologs with regard to the plane PQ; therefore this plane is a symmetry plane of the polyhedron.

Statement XXII. *Every polyhedron, which has an axis of even order and a symmetry plane, which lies normal to this axis, has a symmetry centre, which lies in the intersection point of the axis with the plane.*

Let S', Fig. 2, be the opposite homolog to the vertex S with regard to the axis CA. Let s be the homolog of S' with regard to the plane PQ normal to the axis CA. The two vertices S and s will be homologous with respect to the point C, in which axis and plane meet each other; this point must therefore be a symmetry centre of the polyhedron.

Statement XXIII. *Every polyhedron, which possesses an axis of even order and a symmetry plane, which goes through this axis, has at the same time a second symmetry plane, which passes through this axis and normal to the former plane.*

Let $C\Lambda$, Fig. 7, be the axis of even order, $LC\Lambda$ the given symmetry plane, S a vertex of the polyhedron, S' its homolog on the other side of $C\Lambda$ and s' the homolog of S' with respect to the plane $LC\Lambda$. S and s' will be homologous with respect to the plane $\Lambda CL'$, which passes through $C\Lambda$ and lies normal to the previous one; consequently etc.

Statement XXIV. *In every polyhedron, which possesses an axis L^{2q}, must, if binary, exist axes normal to L^{2q}, whose total number is equal to $2q$, namely q axes of a first kind and q axes of a second kind, which alternate with the first one.*

Let CP, Fig. 5, be a binary axis, which stands normal to the axis CA of the order $2q$; there will be in the plane PCP' normal to CA the straight line CP', which with CP makes the angle:

$$PCP' = \frac{360^\circ}{2q} = \frac{180^\circ}{q}.$$

The straight line CP' will be a binary axis of the same kind as CP, on account of the symmetry coming from the axis CA (Statement X, Addendum). The number of the axes thus obtained, [36] will be equal to q; each one of them can be taken as derivable from CP through the revolution

$$0^\circ, \frac{180^\circ}{q}, 2\frac{180^\circ}{q}, \ldots, (q-1)\frac{180^\circ}{q};$$

the axes, which contain the revolutions

$$q\frac{180^\circ}{q}, (q+1)\frac{180^\circ}{q}, \ldots, (2q-1)\frac{180^\circ}{q},$$

coincide with the former ones.

If one now divides PCP' by the bisecting Cp into two equal parts, then Cp will also be a binary axis. Because if s be the homolog of S with respect to CP, s' be the homolog of s with respect to the axis CA, then the revolution angle, which moves s into s' will be

$$\frac{360^\circ}{2q} = PCP',$$

the vertices S and s' be homologous with respect to Cp, because this will be seen as a binary axis. These q bisecting straight lines will therefore be binary axes, but of another type.

There exist from this $2q$ binary axes, and their number can not be greater; because if there were Q binary axes, and Q were greater than $2q$, then the symmetry order of the axis normal to their plane would be Q or mQ (Statement XIII), which goes against the above conditions.

Addendum I. The number of the binary axes, which are normal to a major axis Λ^{2q}, will always be $= 0$ or $2q$.

Addendum II. The binary axes normal to Λ^{2q} stand pairwisely perpendicular to each other.

Statement XXV. *If in a polyhedron, which possesses an axis L^{2q}, symmetry planes passing through this axis exist, then their total number will be $2q$ and in deed q of one kind and, alternatively, q of another kind.*

[37] Let ACP, ACP', Fig. 5, be two symmetry planes, which make with each other the angles

$$PCP' = \frac{360^\circ}{2q} = \frac{180^\circ}{6}.$$

The system of this plane will be composed of q symmetry planes, which are of the same kind and directly different among one another. Moreover the planes lying among these, which divide the face angle (ACP, ACP') into two equal parts, are also symmetry planes. Because if ACp be one of the planes lying among these, if σ be the homolog of a given vertex S with respect to the plane ACP and σ' the homolog of σ with respect to CA. Here the revolution angle, which brings σ towards σ', takes on the values

$$\frac{360^\circ}{2q} = PCP',$$

therefore it is clear, that S and σ' lie symmetrical with respect to the plane ACp; therefore this plane is a symmetry plane, and there are in total q of which are of a type other than that of ACP, which are all directly different from ACp.

There are therefore $2q$ symmetry planes, which pass through L^{2q}, and their number can not be greater; because if it were equal to $Q > 2q$, then the symmetry order of the axis lying in their common intersection would be equal to Q or mQ (Statement XI), which is not possible.

Addendum I. The number of the symmetry planes, which pass through a major axis Λ^{2q}, is always 0 or $2q$, and can not be greater than this last number.

Addendum II. These symmetry planes stand pairwisely perpendicular to one another.

Statement XXVI. *The polyhedron with major axis Λ^{2q}, which neither possess symmetry planes passing through this axis, nor binary axes, have two different kinds of symmetry, depending on whether the plane normal to the major axis is a symmetry plane or not.*

The number of the symmetry axes is determined by the contents of the statement. In exactly the same way [is] that of the symmetry planes, as soon as one knows, whether the [38] plane normal to the major axis is a symmetry plane or not. Whatever be the case with the existence of a symmetry centre, also hold true the same regarding a symmetry plane normal to the major axis hold (Statement XXI and XXII).

The symbol of these two kinds of symmetry will therefore be:

$$\left[\Lambda^{2q}, 0L^2, 0C, 0P\right]$$

and $$\left[\Lambda^{2q}, 0L^2, C, \Pi\right].$$

Statement XXVII. *The polyhedra with major axis Λ^{2q}, which either have the $2q$ symmetry planes or only the $2q$ binary axes, which are connected with this major axis, can neither have a symmetry plane normal to the major axis, nor a symmetry centre.*

This is an evident consequence of Statements XVII, XVIII, XIX and XX.

The group of the polyhedra, on which the contents of the present Statement visit, falls into two classes of different symmetry, depending on whether the symmetry plane or the binary axes are absent. Here the binary axes of the two different kinds are (Statement XXIV), therefore we wish to indicate those of the first kind with L^2 and those of the second with L'^2. In the same fashion the symmetry planes shall be indicated with P, those of the other type with P'. One has from this, to represent the two classes of the polyhedron, the symbol:

$$\left[\Lambda^{2q}, qL^2, qL'^2, 0C, 0P\right]$$

and $$\left[\Lambda^{2q}, 0L^2, 0C, qP, qP'\right]$$

Statement XXVIII. *In the polyhedra, which have the major axis $\Lambda^2 q$, further $2q$ binary axes and $2q$ symmetry planes passing through the major axis, the binary axes can be placed in symmetry planes, or alternate with them, ie. coincide with the bisection lines of their face angles. This gives for this particular case two different kinds of symmetry.*

For any other position of the axes and planes one against another the binary axes, as they bring forth anew the $2q$ symmetry planes through revolution of 180° (Statement X, Addendum), [39] would double the number of these last ones, and in the same way the number of the binary axes would double, since they would repeat themselves on the other side of the planes in homologous positions (Statement VIII, Addendum), which contradicts the addenda of XXIV and XXV.

Statement XXIX. *In the polyhedra with major axis Λ^{2q} there is, if the symmetry planes contain the binary axes, one symmetry plane normal to the major axis and one symmetry centre.*

Let CA, Fig. 5, be the major axis, CP one of the binary axes and $ACPQ$ the symmetry plane, which contains this axis. The vertex S will have its homolog with regard to the binary axis CP in s; s will have its homolog with regard to the plane $ACPQ$ in Σ. The relative position of S and Σ points to each other, that the plane PCP', normal to CA and to the plane CAP, is a symmetry plane of the polyhedron, with regard to which C and Σ are homologous vertices; therefore a symmetry plane normal to Λ^{2q} will exist, which by itself brings the existence of a symmetry centre with it (Statement XXII).

Statement XXX. *If in the polyhedra with major axis Λ^{2q} the symmetry planes alternate with the binary axes, then the latter are all of the same kind, but each inversely different from the neighbouring ones; then either a symmetry normal to the axis exists, or else a symmetry centre.*

If asymmetry plane normal to the axis exists, then its intersection lines with the symmetry planes would be binary axes (Statement XVII), which contradicts the contents of our Statement; therefore either a symmetry plane normal to the major axis exists, or a symmetry centre does (Statementt XXI).

The neighbouring binary axes $L_0 l_0$ and $L_1 l_1$, Fig. 8, are inversely different from each other, homolog with regard to the symmetry plane lying in between, which goes through the major axis and through $P_0 C p_0$.

The inverse difference in general does not necessarily imply the direct difference; but in the present case it is easy, to be convinced, that $L_0 l_0$ and $L_1 l_1$ can never be directly different, because the revolution, which would lead to the characteristic agreement for direct difference, can realise in nature only either around the major axis or around the angle bisector $p_0 C P_0$ [40] or lastly around the second angle bisector $p_3 C P_3$ normal to the former one. Here $2q$ binary axes exist, therefore one has:

$$L_0 C L_1 = \frac{180^\circ}{2q} = \frac{360^\circ}{4q} \qquad \text{and} \qquad P_0 C P_1 = \frac{180^\circ}{2q} = \frac{90^\circ}{q}.$$

If CL_0 and CL_1 were directly different with respect to the major axis, then in this case the revolution, which reproduces the position of the vertices, would take on the value $\frac{360^\circ}{4q}$, the symmetry order of the major axis $4q$, which is impossible. In the same way CL_0, CL_1 can not be directly different with regard to CP_0, since CP_0 is no binary axis. The normal to CP_0 ie. CP_3 is likewise at least a binary axis, since

$$90^\circ = q \times P_0 C P_1$$

holds, and since then this normal is a straight line of the groups CP_0, CP_1, CP_2. Therefore the neighbouring binary axes are inversely different, without any possibility exists, to bring them to agreement.

Fig. 8 shows the division of the homologous vertices for the case $2q = 6$. The symmetry plane, which lies normal to the major axis, is chosen as plane of the drawing. The filled-up small circles indicate the vertices, which lie under this plane, the ones left white those vertices, which lie above the same plane.

With the convention of the symbol for the two kinds of the symmetry sought after now is to notice, that in the case of Statement XXIX the symmetry planes, stand normal to the binary axes and that in the case, where the faces alternate with the axes, they can not be normal to them. We will therefore, 1. for the case of the coincidence, 2. for the case of the alternates referentially have the formulae:

$$\left[\Lambda^{2q}, qL^2, qL'^2, C, \Pi, qP^2, qP'^2\right]$$

$$\text{and} \quad \left[\Lambda^{2q}, 2qL^2, 0C, 2qP\right].$$

One sees from Statements XXVI, XXVII and XXVIII, that the polyhedra with major axis of even order can have only six kinds of symmetry. One will find them enumerated in the synoptical table at the end of this article.

[41] In this table one can give q all possible values, from $q = 1$ inclusively to $q = \infty$.

Polyhedra with major axis of odd order

Statement XXXI. *A polyhedron, which possesses a major axis of odd number, can not possess at the same time a symmetry plane normal to this axis and a symmetry centre.*

This is a consequence of Statement IV.

Statement XXXII. *If in a polyhedron, which possesses a major axis Λ^{2q+1}, there exist yet other axes, then these are binary, their total number is equal to $2q + 1$, and they are all of the same kind.*

Any one of the new axes, which necessarily is binary, and lies in the plane normal to Λ^{2q+1} (Statement XV), repeats itself $2q$ times, conformably with the Addendum to Statement X; moreover these $2q + 1$ axes will be different among themselves and not pairwisely coincide (as that which happens in the case where the major axes are of even order). In deed, if one enumerate them with 0, 1, 2, etc. in sequence, as revolution

following one after another, always let themt ake the values of $\frac{360^\circ}{2q+1}$, then one finds, that the inclinations of 0, 1, 2, 3 etc. towards the axis O are

$$0^\circ, \frac{360^\circ}{2q+1}, \frac{2\cdot 360^\circ}{2q+1}, \ldots, \frac{q\cdot 360^\circ}{2q+1}, \frac{(q+1)\cdot 360^\circ}{2q+1}, \ldots, \frac{2q\cdot 360^\circ}{2q+1},$$

and correspond to different staight lines, since two of these angles can never differ by 180°.

There can arise in the plane normal to the major axis no other binary axes, than those, whose existence we have just proved. Because were Q be their total number and were $Q > 2q+1$, then the order number of the symmetry of the major axis would be Q or mQ (Statement XIII), which contradicts the supposition of a major axis of the order $2q+1$.

Addendum. The number of the binary axes normal to Λ^{2q+1} must always be 0 or $2q+1$.

[42] **Statement XXXIII.** *In any polyhedron, which possesses the major axis Λ^{2q+1}, if at the same time symmetry planes passing through the major axis exist, these must all be of the same kind and their total number $2q+1$.*

One can point out as in the previous Statement: 1. that the $2q+1$ planes, which are each formed by the amount $\frac{360^\circ}{2q+1}$ around the major axis, are of the same kind and directly different among themselves and not pairwisely coincide; 2. that their number can not exceed $2q+1$ on the account of Statement XI.

Addendum. The number of the symmetry planes, which pass through the major axis Λ^{2q+1}, must always be equal to 0 or $2q+1$.

Statement XXXIV. *The polyhedra with major axis Λ^{2q+1}, which either possess symmetry planes passing through this axis, or binary axes, present three different kinds of symmetry, depending on whether they have a symmetry plane, which lies normal to this axis or not, and in this last case, depending on whether they have a symmetry centre or not.*

If the polyhedron possesses a symmetry plane normal to the major axis, then it can have no symmetry centre (Statement XXXI); then, here the axis is a major axis, the symmetry of the polyhedron is fully determined.

If the plane normal to the majorr axis is no symmetry plane, then the same impossibility no longer precedes. In this manner we will, according to our earlier sign, have the three different symbol:

$$\left[\Lambda^{2q+1}, 0L^2, 0C, 0P\right],$$
$$\left[\Lambda^{2q+1}, 0L^2, C, 0P\right],$$
$$\left[\Lambda^{2q+1}, 0L^2, 0C, \Pi\right].$$

Statement XXXV. *Those polyhedra with major axis Λ^{2q+1}, which either possess only the $2q+1$ symmetry planes or only the $2q+1$ binary axes, which are associable with every major axis, can neither have a symmetry plane lying normal to Λ^{2q+1} nor a symmetry centre.*

[43] This is an obvious consequence from the propositions XVII, XVIII, XIX and XX.

If one sticks to the signs already applied, then one finds, that the two classes of polyhedra, on which the present Statement relate, are represented by the following symbols:

$$\left[\Lambda^{2q+1}, (2q+1)L^2, 0C, 0P\right]$$
$$\text{and} \quad \left[\Lambda^{2q+1}, 0L^2, 0C, (2q+1)P\right].$$

Statement XXXVI. *In the polyhedra, which possess the major axis Λ^{2q+1}, further $2q+1$ binary axes and $2q+1$ symmetry planes going through the major axis, the binary axes lie on the symmetry planes or bisect their face angles.*

This Statement can be proved exactly as Statement XXVIII.

Statement XXXVII. *If in the polyhedra with major axis Λ^{2q+1}, the symmetry planes contain the binary axes, then a symmetry plane is normal to the major axis, but no centres of symmetry exist.*

The proof is the same as for Statement XXIX. But the conclusion related to the existence of the symmetry centre no longer hold on account of Statement XXXI.

Statement XXXVIII. *If in the polyhedra with major axis Λ^{2q+1} the symmetry planes alternate with the binary axes, then the latter are all of the same kind and coincide with the normals of the symmetry planes; there exists then a symmetry centre, but no symmetry plane normal to the major axis.*

The binary axes CL_0, CL_1, CL_2, Fig. 9, then divide the half circumference of a circle, which is described around C as middlepoint, into $2q+1$ equal parts. Here this number is an odd number, therefore one of the bisectors of the angle L_0CL_1, L_1CL_2 etc. will be normal to CL_0; consequently it will always givee a symmetry plane normal to the axis CL_0. It is this the plane, whose trace displays on the plane of Fig. 9 the straight line P_1Cp_1, whereby the last plane is supposed normal to the major axis. The simultaneous existence of a symmetry plane and a binary axis, that stands normal to it, brings that of a symmetry centre with it (Statement XXII).

[44] Moreover the binary axes are all directly different with respect to the major axis, and consequently of the same kind. The white and black circles of Fig. 9 show the positions of the homologous vertices. The black circles mean the vertices placed under the drawing surface, the white ones those placed above this plane.

The symbols of the polyhedra, on which Statement XXXVII and XXXVIII relate, will therefore after the representation taken in S. 12 be:

$$\left[\Lambda^{2q+1}, (2q+1)L^2, C, (2q+1)P^2\right]$$

$$\text{and} \quad \left[\Lambda^{2q+1}, (q+1)L^2, 0C, \Pi, (2q+1)P\right].$$

One sees from Statement XXXIV, XXXV and XXXVIII, that the polyhedra with major axis of odd order can have only the seven kinds of symmetry, which are given in the table at the end of this treatise.

One can give q in this table all possible values, from $q = 1$ inclusively to $q = \infty$.

§ IV. Symmetric spheroidical polyhedra

Statement XXXIX. *Every spheroidical polyhedron has at least two axes L^q and $L^{q'}$, whose order numbers are greater than two.*

A spheroidical polyhedra (Definition IX) can not only have a sole symmetry axis, because otherwise, by avoiding the repetition of this axis through the symmetry planes of the polyhedron (Statement VIII, Addendum), these planes would have to contain the only symmetry axis or coincide with its normal plane. This sole axis could then be viewed as a major axis and the polyhedron would not be spheroidical.

There are as a consequent two or more symmetry axes in the spheroidical polyhedra.

Let L^q and $L^{q'}$ next be two axes, whose order numbers q and q' are not smaller than those of the other axes, therefore I maintain, that $q > 2$ and $q' > 2$.

We firstly assume, that one has $q = 2$ and $q' = 2$. The normal to the plane of the [45] axes L^2 and L'^2 will likewise be a symmetry axis $L^{q''}$ (Statement XI), and here $q'' > q$, $q'' > q'$ can not be the case, therefore $q'' = 2$. Moreover L^2 and L'^2 stand perpendicular to each other, otherwise one would have yet a third binary axis on the plane of the axes L^2 and L'^2, and q'' would be greater than 2 (Statement XIII), which is not possible. One has therefore three binary perpendicular axes L^2 ,L'^2, L''^2, and I maintain, that at least one of these axes can be looked at as a major axis.

In deed no other binary axis other than the three axes L^2, L'^2 and L''^2 can thus occur, because any axis diagonal to L^2 would, together with L^2, be another binary axes in the plane determined by itself and L^2 (Statement X, Addendum), and a symmetry axis of order higher than 2 would result (Statement XIII).

In exactly the same way any symmetry plane must pass through one of the three axes $L^2 \ldots$; without this another three homologous binary axes, with regard to this plane would arise, and the total number of the binary axes would become six, which we have already proved as impossible. Let P then be a symmetry plane, which passes through L^2. If a second symmetry plane exists, then it must be normal to P, otherwise their intersection line would be an axis, whose order would be higher than 2 (Statement XI). We will thus see, whether the position of these planes can bee such a one, that the polyhedron has no major axis.

If the plane P neither passes through L'^2, nor through L''^2, then P must go through L^2, otherwise its intersection straight-line with P would introduce a fourth binary axis, which is impossible. This is exactly the same with the other planes P'' and P''', which necessarily would all have to pass through L^2. Therefore in this case the axis L^2 would suffice the demands placed on the major axis, and the polyhedron would not be spherical.

If on the other hand the plane P did not contain only L^2, but yet one of the two binary axes L'^2 and L''^2 (say the axis L'^2), then the plane P', which must pass through the normal to the plane P, ie through L''^2, would contain the axis L^2 or the axis L'^2 (say the axis L^2), with this its intersection straight-line with P makes no fourth binary axis. If there is then yet a third plane P'', then the same must simultaneously be perpendicular to P and P' (compare above); it will therefore pass through L'^2 and L''^2, and no other symmetry plane can exist. In this case either one of the axes L^2, L'^2 and L''^2 [46] can be viewed as a major axis, and the polyhedron is not spheroidical.

Consequently one can not have $q = 2$ and $q' = 2$.

We now assume, that $q > 2$ and $q' = 2$. The two axes L^q, L'^2 will be perpendicular to each other, otherwise the axis L'^2 would force the axis L^q, to repeat itself at least once, and the number q' would be smaller than the order number of the two axes of the polyhedron, which from the assumption made earlier is not possible. On the same ground no further axis can be given outside the plane normal to L^q. Therefore the first condition, that L^q is a major axis, is therefore made sufficient.

Similarly the symmetry planes of the polyhedron, which are subjected to the condition, must not produce anew the axis L^q, necessarily contain L^q or be perpendicular to it. Therefore L^q will be a major axis, and the polyhedron no spheroidical.

It can not be the case, therefore, that $q > 2$ and $q' = 2$, consequently it must be that $q > 2$ and $q' > 2$.

Statement XL. *If in a polyhedron there exist two axes of order higher than the second one, then the polyhedron is necessarily spheroidical.*

Because if the polyhedron possessed a major axis, the other axes would necessarily be binary (Statement XV), which is against the assumption. Consequently the polyhedron is spheroidical.

Definition X. *The Statements XXXIX and XL show, that one can give yet another definition of the spheroidical polyhedra, than the one in S. 13, such that one says, these polyhedra being symmetrical polyhedra with more axes, at least two of which have a symmetry of higher order than the second one. The polyhedra with major axis could then be defined as polyhedra, which possess one or more symmetry axes, at most one of which is of higher [order] than the second order.*

Statement XLI. *In any spheroidical polyhedron with a [47] symmetry axis L^q of higher order than the second order, the total number Q of the axes of the order q, which belong to this polyhedron, must necessarily*

be equal to half of the value, which the number of vertices of a regular hull polyhedron can have, the condition following this suffices: 1. that the centre of its form at the same time be a symmetry centre; 2. that each of its space vertices of q sides be formed.

The axis L^q is necessarily connected with an axis $L^{q'}$, whereby q' is greater than 2 (Statement XXXIX). If one now turns L^q around $L^{q'}$ by an angle $\frac{360^\circ}{q'}$, one will have the location of a second axis of order q, which is different from the original axis L^q. (Statement X, Addendum).

Let then OA and OB, Fig. 10, be these two axes of the order q, which meet in O, the middle point of the form of the polyhedron. From O as centre we describe the sphere of radius 1, which intersects the two axes OA and OB in A and B, and draw the arc AB of a greatest circle. One can always assume:

$$\text{arc}\, AB < 90^\circ \quad \text{or} \quad = 90^\circ.$$

In the opposite case one would introduce the supplement of the angle AOB in the consideration. One can likewise always assume, that OA and OB be chosen such, that their inclination be the overall smallest of all of those, which the axes of order q have towards one another. This established, we turn the polyhedron by $\frac{360^\circ}{q}$ around the axis OB of order q. The point A will come to C: if we connect BC by the arc of the greatest circle, then

$$ABC = \frac{360^\circ}{q}$$

results, and the straight line OC will likewise be an axis of the order q. (Statement X, Addendum). If we draw in the same manner the arc of the greatest circle CD, so that

$$CD = CB = AB \qquad \text{and} \qquad BCD = \frac{360^\circ}{q},$$

then the straight line OD will also be an axis of the order q.

If one turns the polyhedron a second time by $\frac{360^\circ}{q}$ around the axis [48] OC and from B against D, then the effects of this second revolution is that, the point B goes on to D. The point A remains at C. The two revolutions are equivalent to a single revolution around the point M, the pole of the small sphere circle, which passes through the points A, B, C and D†. The second revolution around OB and OC does not alter the apparent position of the vertices of the polyhedron; as little does the single revolution around M, which replaces it, change this apparent position; therefore the straight line OM will be a symmetry axis of the polyhedron, and one sees, that, if one turns the polyhedron around OM by an angle, which amounts to the face angle AMC, the position of the vertices is not altered in the act. Consequently this angle is commensurate with the circumference (Statement II). Therefore the number of the vertices A, B, C, D etc, which is positioned on the circumference of the small circle $ABCD$, is a limited one: consequently these vertices makes a regular, registered polygon, whose side number can be represented by r. One will always be able to assume, that A and B are two neighbouring vertices; one can therefore write:

$$AMB = \frac{360^\circ}{r},$$

a formula, in which the number r necessarily must be greater than 2.

Here AMB and ABC are parts of 360°, therefore the regular spherical polygon $ABCDE$, which repeats itself in $CBC'D''\ldots$ and in $A'ABC'D'\ldots$, covers in the end the whole surface of the sphere. The totality of the points thus contained will here make the vertices of a regular, registered polyhedron, and this regular polyhedron will necessarily be one of those, in which the sides meet together in a number q, to make each of their vertices.

The five regular polyhedra of geometry all, except the regular tetrahedron, have a symmetry centre in the centre of their form; but in that of the sphere registered tetrahedra the angle distance AB between two vertices exceeds the value 90°. This case therefore can not occur, as it is against our construction procedure.

[49] The registered polyhedron, which results from the construction above, will therefore either be the cube, ie correspond to the case $q = 3$, $r = 4$; one has then‡

$$AB = 70^\circ 32', \quad AM = 54^\circ 44';$$

or the regular octahedron, what corresponds to the case $q = 4$, $r = 3$; one has then:

$$AB = 90^\circ, \quad AM = 54^\circ 44';$$

or the regular dodecahedron, which corresponds to the case $q = 3$, $r = 5$; then

$$AB = 41^\circ 49', \quad AM = 37^\circ 23';$$

or the regular icosahedron, which corresponds to the case $q = 5$, $r = 3$; then

$$AB = 63^\circ 26', AM = 37^\circ 23'.$$

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† This pole M lies in the intersection point of the greatest circle arcs BM and CM, which bisect the spherical angle ABC and BCD.

‡ The arcs AB and AM are given by the well known formulae

$$\cos\frac{1}{2}AB = \text{cosec}\,\frac{\pi}{q}\cos\frac{\pi}{r} \quad \text{and} \quad \cos AM = \cot\frac{\pi}{q}\cot\frac{\pi}{r}.$$

Let M now be the number of the vertices of the regular polyhedron thus contained; here each vertex has a homolog, which lies diametrically opposite to it, therefore one sees, that the total number Q of the axes of the order q will be at least equal to $\frac{1}{2}M$. I maintain besides, that $Q > \frac{1}{2}M$ can not be. Because should $Q > \frac{1}{2}M$, then one of the axes of the order q meet the sphere in a point X, lying in the inside of one of the spherical polygons $ABCDE$. One of the angular distances between X and the vertices A, B, C, D would necessarily be smaller than AM and all the more so AB (from the comparative table of the related values of AB and AM), which the assumption of the least inclination of the two axes OA and OB reiterates. It can not be therefore that

$$Q > \frac{1}{2}M,$$

consequently

$$Q = \frac{1}{2}M.$$

[50] Statement XLII. *A spheroidical polyhedron can only have ternary, quaternary or quinary axes, the binary axes not included.*

This follows from the content of the previous Statement. Here L^q is one of the axes of the polyhedron, therefore the number q can only be the number of sides, which can connect one another, to make the vertex of a regular polyhedron; consequently one must have:

$$q = 3 \quad \text{or} \quad 4 \quad \text{or} \quad 5.$$

Statement XLIII. *There are two different groups of spheroidical polyhedron, that, which contain four ternary axes, and that, which contain ten ternary axes.*

We wish by mutually examining the four cases, to whose opposite generation the Q axes L^q goes, and let M always be the number of the vertices of the regular, registered polyhedron, on to which this kind of repetition goes.

In the case of the cube (Statement XLI)

$$q = 3, \quad M = 8, \quad Q = \frac{1}{2} = 4.$$

In the case of the octahedron

$$q = 4, \quad q = 4, \quad M = 6, \quad Q = \frac{1}{2}M = 3.$$

The three quaternary axes, which one thus obtains, are perpendicular to one another, there are thus four ternary axes (Statement XIV), and it can gives no greater number, because the new ternary axes would force the quaternary axes, to repeat themselves, so that $Q > 3$ would result, which is impossible.

In the case of the dodecahedron

$$q = 3, \quad M = 20, \quad Q = \frac{1}{2}M = 6.$$

[51] Let next M, M_0 and M_1, Fig. 12, be three neighbouring vertices of a registered icosahedron. The normal dropped from the centre of the sphere on the side MM_0M_1 is apparently a ternary axis, and here the icosahedron has twenty pairwisely parallel faces, therefore it will have ten ternary axes; but it can have no greater number of these, because $q = 3$ one only finds the two values $Q = 4$ and $Q = 10$ (Statement XLI) consequently etc.

Addendum. We can therefore divide the spheroidical polyhedra into two groups: the *quaterternary* ones with four ternary axes, which are ordered as the four major diagonals of a cube, and the *decemternary* ones with ten ternary axes, which are ordered as the ten major diagonals of a regular dodecahedron.

Quaterternary polyhedra

Statement XLIV. *If one builds a cube, which has as diagonals the four ternary axes of a given quaterternary polyhedron, then the three normals dropped from the centre of the form on the side faces of this cube are, for the polyhedron three axes of the same kind, and have binary or quaternary symmetry.*

One describes around the centre the form of the polyhedron, the intersect point of its four ternary axes, as midpoint, with the unit as radius, a sphere, which the upper ends of our four ternary axes will intersect in A, A_0, A_1 and A_2, Fig. 11. (I have projected the surface of this sphere stereographically on the plane of the greatest circle, whose pole is A; the reader is however requested, therefore to follow the picture, as if he moves on the sphere. The centre point O of which is not shown on the figure.) The registered cube, whose four upper vertices are A, A_0, A_1 and A_2, will have in B_0, B_1 and B_2 the vertices lying diametrically opposite to A_0, A_1 and A_2. $AA_0B_1B_2$, $AA_0B_2A_1$ and $AA_1B_0A_2$ are three spherical quadratic polygons, whose midpoint M_0, M_1 and M_2 are the endpoints of the three normals, which are dropped from the midpoint on the side faces of the registered cube.

One double turn of 120°, namely first around A_2 as pole, from A on to B_0, and secondly around B_0 as pole, from A_2 [52] on to A_1, A goes on B_0 and A_2 goes on A_1, without changing the position of the vertices. This double turn is equivalent to one turn of 180° around the pole M_0. Therefore OM_0 is an axis, whose symmetry is 2 or a multiple of 2. Besides, the symmetry of OM_0 can not be of higher order than 4; otherwise the number of the ternary axes, which are placed around the pole M_0, would exceed 4, which contradicts the starting conditions. Therefore the three perpendicular axes OM_0, OM_1 and OM_2 are binary or quaternary.

Statement XLV. *The quaternary polyhedra, with binary axes perpendicular to one another, can have no other binary axes.*

If another binary axis existed, then it would be able to cut through the upper part of the sphere surface, Fig. 11, only in one of the three points, which the midpoint, are either the curves AA_0, AA_1 and AA_2, or the homologous curves A_0B_1, A_0B_2, A_1B_2, A_1B_0, A_2B_0 and A_2B_1 then for every other position this axis would force it to repeat the ternary axes, which would double its number. If we assume, that G_0, the centre of AA_0, be the endpoint of the new binary axis, then two turns of the polyhedron, firstly of 180° around the pole G_0, secondly of 120° around the pole A_0 in the sense A towards B_1, bring

$$\begin{aligned} A &\text{ to } A_0, \text{ then to } A_0,\\ A_0 &\text{ to } A, \text{ then to } B_1,\\ B_1 &\text{ to } A_1, \text{ then to } A_2,\\ A_2 &\text{ to } B_2, \text{ then to } A. \end{aligned}$$

These two turns, which leave the place of the vertices of the polyhedron unchanged, are equivalent to a single turn of 90° around M_1 in the sense A towards A_0. Then the pole M_1 would thus be the end of a quaternary axis, which contradicts the initial conditions.

Therefore no other binary axis can exist.

Remark. Let S, Fig. 11, be a vertex of the given polyhedron. One can assume, that this axis is found on the surface of the sphere, in which one takes its distance OS to the centre of the form of the polyhedron as unit; its two homologs with reference to the ternary pole A are S' and S''. The system $SS'S''$ will repeat [53] itself in succession of the binary nature of the pole M_0, M_1 and M_2 in $S_0S_0'S_0''$, $S_1S_1'S_1''$ and $S_2S_2'S_2''$. The complete system of the homologs of any one vertex will therefore in the present case be a polyhedron, which one can register to the sphere.

Addendum. From the arrangement of the twelve vertices of this polyhedron it easily turns out, that it has neither symmetry planes nor a symmetry centre, at least in the general case, where the vertex offers no special feature with regard to its position in the inside of the spherical triangle A_0A_2A. (Compare the proofs of the two following statements.)

Statement XLVI. *The quaterternary polyhedra with binary, right-angled axes can either have six symmetry planes, which coincide with the six planes pairwisely connecting the ternary axes, or otherwise three symmetry, which coincide with the three planes pairwisely connecting the binary axes. Any other symmetry planes can not likewisely be so.*

Every other location of a symmetry plane, from those just mentioned, would cause the four ternary axes to repeat themselves, and must consequently be dismissed.

If A_1AM_1, Fig. 11, depicts one of the symmetry planes, then the nature of the pole A demands that A_1AM_2 and A_0AM_0 also be likewise; the ternary nature of the pole A_1 forces us to extend this conclusion to the planes $B_2A_1M_0$ and $B_0A_1M_2$, and the same applies to $A_0M_1A_2$ in view of the binary axis OM (Statement XXIII.)

In this case every one of the triangles $SS'S''$, $S_0S_0'S_0''$, $S_1S_1'S_1''$ and $S_2S_2'S_2''$ is replaced by a hexagon. We have further confined ourselves to portray the one which encloses the vertex B_0. The twenty-four vertices of the polyhedron can be reduced to twelve, if the vertex S, which is viewed as important for the positions of all others, fell on one of the three greatest arcs A_0AM_0, A_1AM_1, A_2AM_2. These vertices can be reduced to four, if S coincided with A etc.

Let us assume, that $M_2G_0M_1$ represents a symmetry plane; [54] then, as a consequence of the ternary nature of the pole A, would also $M_1G_2M_0$ and $M_0G_1M_2$ be symmetry planes. In that case the triangle $SS'S''$ repeats itself around the pole A_0, A_1, and A_2. We have confined ourselves thereupon, to portray this kind of repetition around the pole A_1. The triangle $\sigma\sigma'\sigma''$ is then the homolog of the triangle $SS'S''$ with respect to the symmetry plane $M_2G_1M_0$.

Incidentally these two systems of symmetry planes can not occur at the same time, because it would otherwise give four symmetry planes, which intersect one another at the pole M_0, and the axis OM_0 would then be at least quaternary (Statement XI), which contradicts conditions made in the statement.

Statement XLVII. *The quaternary polyhedra with binary right-angled axes can only have one symmetry centre under the condition, that they have three symmetry planes, which pairwisely join the binary axes, and vice versa the existence of these planes entails the same of the symmetry centre.*

This is a consequence from Statement XXI and XXII.

Statement XLVIII. *The quaternary polyhedra with binary right-angled axes can only have three different kinds of symmetry, depending on whether they have no symmetry planes, or six going through the ternary axes, or three symmetry planes going through the binary axes.*

This is a consequence from the addendum to Statement XLV and from Statement XLVI. In view of Statement XLVII and the definition, which we made S 12, one has the three symbols:

$$\begin{aligned} &\left[4L^3, 3L^2, 0C, 0P\right],\\ &\left[4L^3, 3L^2, C, 3P^2\right],\\ &\left[4L^3, 3L^2, 0C, 6P\right]. \end{aligned}$$

Statement XLIX. *Every quaterternary polyhedron with quaternary axes has six binary axes, which the opposite edges of a cube, of which the four ternary axes of the polyhedron are diagonals, pairwisely join.*

[55] I maintain, that, if one connect the midpoint O of the sphere, Fig. 11, with the point G_0, the middle of AA_0, this straight line must be a binary axis of the polyhedron.

If we give the polyhedron one turn of 90° around OM_1 from A into A_0, and a second turn of 120° around OA_0 from B_1 into A, then these double movements, which do not change the apparent position of the vertices, go from

A to A_0, then to A_0,
A_0 to B_1, then to A,
B_1 to A_2, then to A_1.
A_2 to A, then to B_2.

The result of these two movements is the same, as if one had turned the polyhedron 180° around G_0; thus OG_0 is an axis of even order, that obviously can only be binary, and the same would be the case for the five remaining straignt lines, the homologs of OG_0. Three of the six binary axes are situated in the plane of the base circle of the projection of the sphere.

One could prove, as has already been done in the proof of Statement XLV, that every other straight line would be unsuitable, to be a binary axis of the system.

Statement L. *The quaterternary polyhedra with quaternary axes have, if they in general have symmetry planes, of which necessarily six, which pass through the ternary axes, and three, which pass through the quaternary axes; they have at the same time a symmetry centre.*

I will indicate the planes, which pass through the quaternary axes with P^4 and those, which pass through the ternary axes, with P^2. One has already earlier seen, (with the proof of Statement XLVI), that these are the only possible symmetry planes.

If we accept the existence of the planes P^4; $M_0G_1M_2$ and $M_0G_2M_1$, Fig. 11, will be two symmetry planes, which meet each other in a quaternary axis, consequently AM_0B_0 and $A_1B_0A_2$ will likewise be symmetry axes (Statement XXV). The system P^2 therefore joins the system P^4. One would prove in exactly the same way, that the system of planes P^2 always presupposes the coexistence of the system of planes P^4.

Each of the three planes of the system P^4 is normal to one of the three quaternary axes. [56] The six planes of the system P^2 are each normal to one of the six binary axes; because if in a cube with four vertical edges, one joins the centres of two mutually opposing ones of these four edges, then each of these straight lines will be normal to the plane, which passes through the two other edges. The existence of the symmetry planes furthermore entails that of the symmetry centre (Statement XXII).

Statement LI. *With the quaterternary polyhedra with quaternary axes are only admissible two kinds of symmetry, depending on whether they have symmetry planes or not.*

This follows from the previous Statement.

In the case, where the polyhedron has no symmetry plane, it can have no symmetry centre, as consequence of Statement XXI. The complete system of the homologs of the vertex S, Fig. 11, builds then a registered polyhedron of twenty-four vertices, which are grouped into three's around each one of the eight poles A, A_0, A_1, A_2, B_0 etc. One has with the Figure confined oneself to showing the triangle $\sigma_2\sigma_2'\sigma_2''$, which surrounds the pole A_2 in this case.

If the polyhedron has its nine symmetry planes, then the eight triangles $SS'S''$, $S_0S_0'S_0''$ etc. are replaced by eight hexagons, and the system of the homologs contains fourty-eight vertices.

The symbol of these two kinds of symmetry are the following:

$$\left[3L^4, 4L^3, 6L^2, 0C, 0P\right]$$
$$\left[3L^4, 4L^3, 6L^2, C, 3P^4, 6P^2\right].$$

Decemternary polyhedra

Statement LII. *If one constructs a regular dodecahedron which has as diagonals the ten ternary axes of a given decemternary polyhedron, then the six normals which fall from the centre of the form on the side faces of this dodecahedron are quinary symmetry axes for this polyhedron.*

With the unit as radius, and the common intersection of the ten ternary axes as midpoint, one describes a sphere which in A_0, A_1, [57] A_2, A_3, A_4, B_0, B_1, B_2, B_3, B_4, Fig. 12, intersects the upper halfs of the ten ternary axes. (I have stereographically projected the surface of this sphere on the plane of the greatest circle which lies parallel to the face $A_0A_1A_2A_3A_4$ of the registered regular dodecahedron whose vertices are these end points A_0, A_1, ..., B_0[,] B_1 etc. The reader is asked to imagine the points and lines of the figure on the surface of the sphere; the midpoint O of the sphere is not given on the drawing.) By pairwisely connecting the intersection point, regular spherical pentagons result whose twelve midpoints M, M_0, M_1, ..., N_0, N_1 etc. are the extreme ends of the same radii which stretches from the midpoint of the sphere normal to the twelve side faces of the dodecahedron.

Two turns of 120°, one around OA_0, from A_4 towards B_0, the other around OB_0 from A_0 towards C_2, A_4 will go on to B_0 and A_0 on to C_2. These two turns, which do not differ from the position of the vertices of the polyhedron, are of the same value with an only turn of 144° around OM_2 from A_0 towards B_0; this would, if repeated three times, be the same as one turn 72°; therefore the axis OM_2 is a quinary axis, and the same holds with OM, OM_0, OM_1, etc. The points M_0, M_1, etc. are the vertices of a regular icosahedron described by the sphere.

Statement LIII. *The decemternary polyhedra always have fifteen binary axes.*

One turns the given polyhedron 72° around OM, Fig. 12, from A_0 onto A_1, then 120° around OA, from A_2 onto A_0; as consequence of these two turns the pole A_0 comes over A_1 and A_1 comes over A_0. The end result is the same, as if the polyhedron were turned 180° around the radius OG, which ends in the middle of the curve A_0A_1. Now the apparent places of the vertices are not changed; consequently G is the end point of an axis of even order, that is obviously a binary axis. Here the regular dodecahedron has thirty pairwisely opposite sides, therefore the total number of binary axes is equal to fifteen.

No other diameter of the sphere can be a binary axis, because which ever one likes to be in its position, therefore it would force itself to repeat the ternary axes, and one would obtain more than ten ternary axes, that which we know is impossible. (Statement XLIII).

[58] One could also obtain the fifteen binary axes, in which one pairwisely joins the centres of the edges facing one another of the icosahedron MM_0M_1 ....

Statement LIV. *The decemternarial ppolyhedra can have as symmetry planes the fifteen planes, which go through the six quinarial axes pairwisely joined. In the opposing cases these polyhedra have no symmetry plane at all.*

Let us take a look specially at a plane, which goes through the centre of the sphere and through the vertices M and M_3, Fig. 12.

This plane becomes a symmetry plane for the point system (A_0, A_1), (A_4, A_2), ..., (M_2, M_4), M_1, M_0 etc.; it therefore does not result in the doubling of the number of axes, consequently it answers nothing about the existence of such a symmetry plane.

Now lies the plane MM_3 normal to the straight line KK', which is a binary axis of the system. The homologs of this plane are in total fifteen in number, namely MM_0, MM_1, MM_2, MM_3, MM_4; M_0M_2, M_1M_3, M_2M_4, M_3M_0, M_4M_1; M_0M_1, M_1M_2, M_2M_3, M_3M_4, M_4M_0. These are obviously the only symmetry planes, which the polyhedron can have; for any other position the number of the ternary axes would exceed 10, which can not be the case. (Statement XLIII)

Figure 12 shows the arrangement of the sixty homologous vertices of S around the pole A_0, A_1, etc. in the case where the polyhedron has no symmetry plane at all.

If the fifteen symmetry planes, which are mentioned above, exist in the polyhedron, then the triangle $SS'S''$ is replaced by a hexagon. For the figure not to be overcrowded, one has restricted oneself to showing only one of the hexagons, the one which would enclose the pole B_0. The system of the homologs of S then contains hundred and twenty vertices. For certain special positions of S this number can be reduced to sixty, to twenty, or to twelve vertices. This last case occurs when the vertice S is placed at the end of one of the quinary axes of the system.

Statement LV. *When the decemternary polyhedron has the fifteen symmetry planes, which are given in the previous statement, then these planes are [59] normal to the fifteen binary axes, and the polyhedron has a symmetry centre. In the contrariwise case it has none.*

It follows from the proof of the statement above, that the binary axis KOK', Fig. 12, is normal to the plane M_3GMA_3. Now this plane is actually one of the fifteen symmetry planes of the polyhedron, consequently each one of these planes is normal to one of teh fifteen binary axes, therefore the polyhedron then has a symmetry centre (Statement XXII). If the polyhedron were without symmetry planes, then it would follows that its binary axes have no symmetry centre. (XXI)

Statement LVI. *The decemterary polyhedra have two different kinds of symmetry, depending on whether it has a symmetry centre or not.*

This is a consequence of Statements LIV and LV.

The symbols of these two types are:

$$\left[6L^5, 10L^3, 15L^2, 0C, 0P\right],$$
$$\left[6L^5, 10L^3, 15L^2, C, 15P^2\right].$$

Statement LVII. *The axes, which characterise the symmetry of the quaterternary polyhedra with binary right-angled axes, also enter into the symmetry of the decemternary polyhedra.*

Let us choose an arbitrary point of a ternary axis, eg. A_2, Fig. 12. The centre G of one of the two sides A_1A_0, A_3A_4, the side A_0A_4 lying opposite to the vertex A_2 in the pentagon $A_0A_1A_2A_3A_4$ of which closely fit, will be the endpoint of a binary axis (Statement LIII). The same will be the case with the points H and K, which are the homologs of G with regard to the ternary axis OA_2. In the spherical triangle HGK the three angles H, G and K are right [angles]. Therefore the three sides HG, KG and HK are equal to $90°$.

The three axes OG, OH and OK are consequently three right-angled binary axes, and the spherical triangle GHK is such a one with three right angles. The vertex A_2 is the midpoint of this triangle. In exactly the same way A_4 will be the mispoint of the triangle GHK' likewisely possessing three right angles, B_0 that [60] of the triangle $GK'H'$ obtained in exactly the same way, when H' is the lower end of the axis OH, and B_1 the midpoint of the triangle KGH' with the three right angles.

The four ternary axes OA_2, OA_4, OB_0, OB_1 then combine to one another with the three binary right-ngled axes in the same position, which characterises the quaterternary polyhedra with binary right-angled axes.

Remark. One could replace the combination OA_2, OA_4, OB_0, OB_1 with one of the following four combinations.

$$[OA_0, OA_3, OB_1, OB_2], \qquad [OA_0, OA_2, OB_3, OB_4]$$
$$[OA_1, OA_3, OB_0, OB_4], \qquad [OA_1, OA_4, OB_2, OB_3].$$

Statement LVIII. *The polyhedra* $\left[6L^5, 10L^3, 15L^2, 0C, 0P\right]$ *have all elements of the symmetry of the polyhedra* $\left[4L^3, 3L^2, 0C, 0P\right]$. *The polyhedra* $\left[6L^5, 10L^3, 15L^2, C, 15P^2\right]$ *have all elements of the symmetry of the polyhedra* $\left[4L^3, 3L^2, C, 3P^2\right]$.

The part relating to the symmetry axes of this statement has already been proved in the previous Statement. One concludes from this easily, that the polyhedra $\left[6L^5, 10L^3, 15L^2, 0C, 0P\right]$ have all elements of the symmetry of the polyhedra $\left[4L^3, 3L^2, 0C, 0P\right]$.

If the decemternary polyhedron [has] fifteen symmetry planes besides, then the face KG, GH, HK of Fig. 12 will be below it, and the planes $3P^2$ will represent the polyhedron $\left[4L^3, 3L^2, C, \ 3P^2\right]$. Here the centre of the symmetry C in both cases exists, therefore one sees, that the symmetry characterised by $\left[4L^3, 3L^2, C, 3P^2\right]$ is included with the polyhedra $\left[6L^5, 10L^3, 15L^2, C, 15P^2\right]$ complete.

Remark. The theorem LVII and LVIII are for the general theory of the symmetrical polyhedra only indirectly imortant. They are here taking into consideration the application given, which one can make of in the crystallography with the study of regular system.

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